MORSE METHOD IN COMPUTING PERSISTENT COSHEAF HOMOLOGY

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ABSTRACT. The main aim of this essay is to use the result from discrete Morse theory to simplify the computation of cosheaf homology and persistent cosheaf homology. Throughout this essay I will adopt a categorical language to obtain a much clearer picture of what we do and why we do them. In the end, I address a drawback and a possible solution to the current framework.

CONTENTS

1. THE CATEGORY OF COSHEAVES

In this section, we describe what is the category of cosheaves of a simplicial complex K, which will be denoted as $\mathrm{Cosh}(K)$. In particular, we will describe the abelian category structure on $\mathrm{Cosh}(K)$, and also study how to take limits and colimits, hence direct sums, kernels, cokernels, etc., in $\mathrm{Cosh}(K)$.

Definition 1.1 (Category of Cosheaves)**.** *Given a simplicial complex K, a cosheaf (of vector spaces based on the field $\mathbb{F}$) $\mathcal{F}$ on K is a contravariant functor*

$$\mathcal{F} : (K, \leq) \to \mathbf{Vect}_{\mathbb{F}}$$

Or equivalently, it is a covariant functor from $(K, \leq)^{\mathrm{op}}$ to $\mathbf{Vect}_{\mathbb{F}}$. For each simplex $\sigma \in K$, $\mathcal{F}(\sigma)$ is called the stalk of $\mathcal{F}$ at σ. Cosheaves on K form a category $\mathrm{Cosh}(K)$, which by definition is the following functor category,

$$\mathrm{Cosh}(K) := [(K, \leq)^{\mathrm{op}}, \mathbf{Vect}_{\mathbb{F}}]$$

Explicitly, a morphism from a cosheaf $\mathcal{F}$ to a cosheaf $\mathcal{G}$ is a natural transformation

$$\eta : \mathcal{F} \to \mathcal{G}$$

η consists of a family of maps η_σ indexed by $\sigma \in K$, such that for any two simplices σ, τ that $\sigma \leq \tau$, the following diagramme commutes,

$$\begin{array}{ccc} \mathcal{F}(\tau) & \xrightarrow{\mathcal{F}(\sigma\leq\tau)} & \mathcal{F}(\sigma) \\ {\scriptstyle \eta_\tau}\downarrow & & \downarrow{\scriptstyle \eta_\sigma} \\ \mathcal{G}(\tau) & \xrightarrow[\mathcal{G}(\sigma\leq\tau)]{} & \mathcal{G}(\sigma) \end{array}$$

Functor categories have very nice properties, and those particularly concern us are the two lemmas below.

Lemma 1.2. *For any category $\mathcal{C}$ and any (co)complete category $\mathcal{D}$, the category $[\mathcal{C}, \mathcal{D}]$ is (co)complete, and the (co)limit of a diagramme $I \to [\mathcal{C}, \mathcal{D}]$ is computed point-wise. Explicitly, for any $x \in \mathcal{C}$, we have the following natural isomorphism*

$$(\varprojlim F_i)(x) \cong \varprojlim F_i(x)$$

if $\mathcal{D}$ is complete and

$$(\varinjlim F_i)(x) \cong \varinjlim F_i(x)$$

if $\mathcal{D}$ is cocomplete. In the above two formulas the (co)limit on the left is computed in $[\mathcal{C}, \mathcal{D}]$, and the (co)limit on the right is computed in the $\mathcal{D}$.

For a proof of Lemma 1.2 see [2]. In particular, Lemma 1.2 shows that both products, coproducts, equalizers and coequalizers are computed point-wise in a functor category, which is very useful when we do computations in $\mathrm{Cosh}(K)$.

The other lemma indicates that if $\mathcal{A}$ is an abelian category, then the functor category from any other category $\mathcal{C}$ to $\mathcal{A}$ inherits a canonical abelian structure.

Lemma 1.3. *If $\mathcal{A}$ is an abelian category, then so does the functor category $[\mathcal{C}, \mathcal{A}]$ for any category $\mathcal{C}$.*

The same as the case for limits and colimits, the abelian structure on $[\mathcal{C}, \mathcal{A}]$ is defined point-wise using the abelian structure on $\mathcal{A}$. For any $F, G : \mathcal{C} \to \mathcal{A}$, let $\eta, \epsilon \in \mathrm{Hom}_{[\mathcal{C},\mathcal{A}]}(F, G)$. Their addition $\eta + \epsilon$ is then given by a family of maps

$$(\eta + \epsilon)_x := \eta_x + \epsilon_x, \ \forall x \in \mathcal{C}$$

The direct sum of F and G is again defined point-wise by the direct sum in $\mathcal{A}$,

$$(\mathcal{F} \oplus \mathcal{G})(x) := \mathcal{F}(x) \oplus \mathcal{G}(x)$$

To verify the above data gives a well-defined abelian structure on $[\mathcal{C}, \mathcal{A}]$ is not that relevant to our goal in this essay, and is not very hard essentially. Hence, I will omit the proof here. I just want to further remark that many abelian categories are special cases of Lemma 1.3. If C is the discrete category of natural numbers then $[C, \mathbf{Ab}]$ is the category of graded abelian groups; if $(\mathbb{N}, \leq)$ is the category of natural numbers with the smaller than or equal to relation, then the category $[(\mathbb{N}, \leq), \mathbf{Ab}]$ is the category of persistent modules of abelian groups; for any abelian category $\mathcal{A}$, the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes on $\mathcal{A}$ is a full subcategory of $[(\mathbb{N}, \leq), \mathcal{A}]$.

The abelian structure in all these categories are given point-wise. See [1] for more discussions. What related to us is that $\mathrm{Cosh}(K)$ is abelian:

Corollary 1.4. *For any simplicial complex K, the category $\mathrm{Cosh}(K)$ is abelian.*

Proof. It is well-known that $\mathbf{Vect}_{\mathbb{F}}$ is an abelian category, then by Lemma 1.3 it follows that $\mathrm{Cosh}(K)$ as the functor category $[(K, \leq)^{\mathrm{op}}, \mathbf{Vect}_{\mathbb{F}}]$ is also abelian. □

Hence, it is legitimate to talk about kernels, cokernels, images, etc., in $\mathrm{Cosh}(K)$, and Lemma 1.2 shows us how to compute them:

Proposition 1.5. *For any morphism $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Cosh}(K)$ and any $\sigma \in K$ we have*

$$(\ker \eta)_\sigma = \ker \eta_\sigma, \; (\mathrm{coker}\, \eta)_\sigma = \mathrm{coker}\, \eta_\sigma, \; (\mathrm{img}\, \eta)_\sigma = \mathrm{img}\, \eta_\sigma$$

Proof. By definition, $\ker \eta$ is the equalizer of η and 0, which is a limit construction. By Lemma 1.2, limit are computed point-wise, thus we have

$$(\ker \eta)_\sigma = \ker \eta_\sigma$$

Similarly, cokernel of η is the coequalizer of η and 0, which is a colimit construction. Hence it follows

$$(\mathrm{coker}\, \eta)_\sigma = \mathrm{coker}\, \eta_\sigma$$

Finally, $\mathrm{img}\, \eta$ is defined as the kernel of the cokernel,

$$\mathrm{img}\, \eta = \ker(\mathcal{G} \to \mathrm{coker}\, \eta)$$

Then applying Lemma 1.2 again results in

$$(\mathrm{img}\, \eta)_\sigma = \mathrm{img}\, \eta_\sigma$$

The above shows both kernels, cokernels and images are computed point-wise. □

Corollary 1.6. *A morphism $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Cosh}(K)$ is injective iff η_σ is injective for any $\sigma \in K$.*

Proof. In any abelian category, a morphism is injective iff its kernel is zero. Now by Proposition 1.5, we know that

$$\ker \eta = 0 \Leftrightarrow \ker \eta_\sigma, \; \forall \sigma \in K$$

It follows that η is injective iff η_σ is injective for any $\sigma \in K$. □

Another consequence of the fact that all the kernels and cokernels are computed point-wise is that exactness at cosheaf level implies exactness at all stalks.

Proposition 1.7. *Given a sequence in $\mathrm{Cosh}(K)$,*

$$\mathcal{F} \xrightarrow{\eta} \mathcal{G} \xrightarrow{\epsilon} \mathcal{H}$$

if it is exact then for each $\sigma \in K$, the following is an exact sequence in $\mathbf{Vect}_{\mathbb{F}}$,

$$\mathcal{F}(\sigma) \xrightarrow{\eta_\sigma} \mathcal{G}(\sigma) \xrightarrow{\epsilon_\sigma} \mathcal{H}(\sigma)$$

Proof. η and ϵ is exact at $\mathcal{G}$ iff the following holds,

$$\mathrm{img}\, \eta = \ker \epsilon$$

Now since by Proposition 1.5, both image and kernel are computed point-wise, it follows that we require

$$\operatorname{img} \eta_\sigma = \ker \epsilon_\sigma$$

for any $\sigma \in K$, which exactly states the exactness of η_σ and ϵ_σ for every $\sigma \in K$. $\square$

Proposition 1.7 will be used in the next section when we prove the exactness of the functor of associated chain complex of cosheaves.

2. Functor of Associated Chain Complex of Cosheaves

The main goal of this section is to construct and study the functor which associates to each cosheaf on K to a chain complex on $\mathbf{Vect}_{\mathbb{F}}$,

$$\mathbf{C}_\bullet(K; -) : \operatorname{Cosh}(K) \to \mathbf{Ch}(\mathbf{Vect}_{\mathbb{F}})$$

In other words, it takes in a cosheaf $\mathcal{F}$ on K and outputs a chain complex $\mathbf{C}_\bullet(K; \mathcal{F})$ on $\mathbf{Vect}_{\mathbb{F}}$; and takes a morphism of cosheaves $\eta : \mathcal{F} \to \mathcal{G}$ to a chain map

$$\mathbf{C}_\bullet(K; \eta) : \mathbf{C}_\bullet(K; \mathcal{F}) \to \mathbf{C}_\bullet(K; \mathcal{G})$$

The reason we are interested in such a functor is that it allows us to turn a filtration of cosheaves

$$\mathcal{F}_1 \hookrightarrow \mathcal{F}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_n$$

into a sequence of chain complexes

$$\mathbf{C}_\bullet(K; \mathcal{F}_1) \to \mathbf{C}_\bullet(K; \mathcal{F}_2) \to \cdots \to \mathbf{C}_\bullet(K; \mathcal{F}_n)$$

and hence allows us to define the persistent homology associated to the filtration of cosheaves.

To simplify our discussions in the future, let's first introduce some notations:

Definition 2.1. *Assume the vertices of K is well-ordered. Given a pair of simplices $\sigma, \tau \in K$, the boundary coefficient $[\sigma : \tau]$ is given by*

$$[\sigma : \tau] := \begin{cases} +1 & \sigma = \tau_{-i} \text{ and } i \text{ is even.} \\ -1 & \sigma = \tau_{-i} \text{ and } i \text{ is odd.} \\ 0 & \text{otherwise} \end{cases}$$

If $[\sigma : \tau]$ is not zero, i.e. $\sigma \leq \tau$ and σ has codimension 1 in τ, then we write $\sigma \triangleleft \tau$.

Definition 2.2. *Given any cosheaf $\mathcal{F}$ on K, for any two simplices $\sigma, \tau \in K$, let $\mathcal{F}_{\tau,\sigma}$ denote a map of the following type*

$$\mathcal{F}_{\tau,\sigma} : \mathcal{F}(\tau) \to \mathcal{F}(\sigma)$$

which is defined as follows,

$$\mathcal{F}_{\tau,\sigma} = \begin{cases} [\sigma; \tau]\mathcal{F}(\sigma \leq \tau) & \text{if } \sigma \triangleleft \tau \\ 0 & \text{otherwise} \end{cases}$$

The abbreviations here defined are essentially the same as in the lecture notes, only with perhaps some different specific choices. According to our definition, the map $\mathcal{F}_{\tau,\sigma}$ is none zero iff $\sigma \lhd \tau$; it equals $\mathcal{F}(\sigma \leq \tau)$ if $[\sigma : \tau] = 1$ and $-\mathcal{F}(\sigma \leq \tau)$ otherwise.

The following lemma is very useful in future proofs:

Lemma 2.3. *For any $\tau, \omega \in K$ such that $\dim \tau = k+1$ and $\dim \omega = k-1$ where $k \in \mathbb{N}_+$, we have*

$$\sum_{\dim \sigma = k} \mathcal{F}_{\sigma,\omega} \circ \mathcal{F}_{\tau,\sigma} = 0$$

Proof. If $\omega \not\leq \tau$ in K, it is trivially true, because at least one of $\mathcal{F}_{\sigma,\omega}, \mathcal{F}_{\tau,\sigma}$ would be zero. If $\omega \leq \tau$, then suppose

$$\tau = [v_0, \cdots, v_{k+1}]$$

and suppose

$$\omega = [v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_{k+1}]$$

The only k-simplices with a non-zero contribution would be

$$\sigma_i = [v_0, \cdots, \widehat{v_i}, v_{k+1}],\ \sigma_j = [v_0, \cdots, \widehat{v_j}, v_{k+1}]$$

By definition,

$$\mathcal{F}_{\sigma_i,\omega} \circ \mathcal{F}_{\tau,\sigma_i} = [\omega : \sigma_i][\sigma_i : \tau]\mathcal{F}(\omega \leq \tau) = (-)^{j-1}(-)^i \mathcal{F}(\omega \leq \tau)$$

$$\mathcal{F}_{\sigma_j,\omega} \circ \mathcal{F}_{\tau,\sigma_j} = [\omega : \sigma_j][\sigma_j : \tau]\mathcal{F}(\omega \leq \tau) = (-)^i(-)^j \mathcal{F}(\omega \leq \tau) = -\mathcal{F}_{\sigma_i,\omega} \circ \mathcal{F}_{\tau,\sigma_i}$$

It follows that

$$\sum_{\dim \sigma = k} \mathcal{F}_{\sigma,\omega} \circ \mathcal{F}_{\tau,\sigma} = \mathcal{F}_{\sigma_i,\omega} \circ \mathcal{F}_{\tau,\sigma_i} + \mathcal{F}_{\sigma_j,\omega} \circ \mathcal{F}_{\tau,\sigma_j} = 0$$

Combining all the above arguments complete the proof. □

Now we can proceed to the construction of the functor $\mathbf{C}_\bullet(K; -)$. The construction of the associated chain complex of a cosheaf $\mathcal{F}$ is essentially the same as the case of sheaves:

Definition 2.4. *For any $\mathcal{F}$ in $\mathrm{Cosh}(K)$, the associated chain complex $\mathbf{C}_\bullet(K; \mathcal{F})$ contains the following data:*

- *for any $k \in \mathbb{N}$, the vector space of k-chains with $\mathcal{F}$ coefficients is the following direct sum,*

$$\mathbf{C}_k(K; \mathcal{F}) := \oplus_{\dim \tau = k} \mathcal{F}(\tau)$$

- *for any $k \in \mathbb{N}$, the boundary map $d_k^{\mathcal{F}}$ is defined block-wise; for any τ, σ that $\dim \tau = k$ and $\dim \sigma = k-1$, the τ, σ component $d_k^{\mathcal{F}}|_{\tau,\sigma} : \mathcal{F}(\tau) \to \mathcal{F}(\sigma)$ is given by*

$$d_k^{\mathcal{F}}|_{\tau,\sigma} := \mathcal{F}_{\tau,\sigma}$$

Proposition 2.5. *For any cosheaf $\mathcal{F}$ on K and any $k \in \mathbb{N}$, we have*

$$d_k^{\mathcal{F}} \circ d_{k+1}^{\mathcal{F}} = 0$$

Proof. Suppose we have two simplices $\tau, \omega \in K$ such that $\dim \tau = k+1$ and $\dim \omega = k-1$, then the component $\mathcal{F}(\tau) \to \mathcal{F}(\omega)$ of $d_k^{\mathcal{F}} \circ d_{k+1}^{\mathcal{F}}$ is given by

$$
\begin{aligned}
(d_k^{\mathcal{F}} \circ d_{k+1}^{\mathcal{F}})|_{\tau,\omega} &= \sum_{\dim \sigma = k} d_k^{\mathcal{F}}|_{\sigma,\omega} \circ d_{k+1}^{\mathcal{F}}|_{\tau,\sigma} \\
&= \sum_{\dim \sigma = k} \mathcal{F}_{\sigma,\omega} \circ \mathcal{F}_{\tau,\sigma} \\
&= 0
\end{aligned}
$$

The final step holds by Lemma 2.3. □

Proposition 2.5 shows that the chain complex $\mathbf{C}_\bullet(K;\mathcal{F})$ we described in Definition 2.4 is a well-defined chain complex. However, the more important observation is that a morphism between cosheaves would induce a chain map between the two associated chain complexes.

Definition 2.6. *Given any morphism $\eta : \mathcal{F} \to \mathcal{G}$ in* $\mathrm{Cosh}(K)$ *and any $k \in \mathbb{N}$, it induces a map between the associated k-th chain groups of $\mathcal{F}$ and $\mathcal{G}$,*

$$\mathbf{C}_k(K;\eta) : \mathbf{C}_k(K;\mathcal{F}) \to \mathbf{C}_k(K;\mathcal{G})$$

which is defined as

$$\mathbf{C}_k(K;\eta) := \oplus_{\dim \sigma = k} \eta_\sigma$$

Matrix-wise, $\mathbf{C}_k(K;\eta)$ is a block-diagonal matrix of the following form,

$$\mathbf{C}_k(K;\eta) = \begin{pmatrix} \eta_{\sigma_1} & & \\ & \ddots & \\ & & \eta_{\sigma_n} \end{pmatrix}$$

where each σ_i has dimension k, and all empty blocks in the above matrix are zero.

Proposition 2.7. *For any morphism $\eta : \mathcal{F} \to \mathcal{G}$ and any $k \in \mathbb{N}$, the following diagramme commutes,*

$$
\begin{array}{ccc}
\mathbf{C}_k(K;\mathcal{F}) & \xrightarrow{d_k^{\mathcal{F}}} & \mathbf{C}_{k-1}(K;\mathcal{F}) \\
\Big\downarrow {\scriptstyle \mathbf{C}_k(K;\eta)} & & \Big\downarrow {\scriptstyle \mathbf{C}_{k-1}(K;\eta)} \\
\mathbf{C}_k(K;\mathcal{G}) & \xrightarrow[d_k^{\mathcal{G}}]{} & \mathbf{C}_{k-1}(K;\mathcal{G})
\end{array}
$$

Proof. Take any $\tau \in K$ such that $\dim \tau = k$. For any $v \in \mathcal{F}(\tau)$, we have

$$
\begin{aligned}
\mathbf{C}_{k-1}(K;\eta) \circ d_k^{\mathcal{F}}(v) &= \sum_{\sigma \lhd \tau} [\sigma : \tau] \eta_\sigma \circ \mathcal{F}(\sigma \le \tau)(v) \\
&= \sum_{\sigma \lhd \tau} [\sigma : \tau] \mathcal{G}(\sigma \le \tau) \circ \eta_\tau(v) \\
&= d_k^{\mathcal{G}} \circ \mathbf{C}_k(K;\eta)(v)
\end{aligned}
$$

The first and third equalities hold by definition, in particular because of $\mathbf{C}_\bullet(K;\eta)$ is block-diagonal. The second equality holds by naturality of η. □

Corollary 2.8. *Any morphism $\eta : \mathcal{F} \to \mathcal{G}$ in* $\mathrm{Cosh}(K)$ *induces a chain map between the associated chain complex of $\mathcal{F}$ and $\mathcal{G}$,*

$$\mathbf{C}_\bullet(K;\eta) : \mathbf{C}_\bullet(K;\mathcal{F}) \to \mathbf{C}_\bullet(K;\mathcal{G})$$

Proof. This is a direct consequence of Proposition 2.7. □

Up to this point, we've described all the data of the functor $\mathbf{C}_\bullet(K;-)$ we intend to define. The following shows it is indeed a functor, which is additive and furthermore is *exact*.

Proposition 2.9. *The data contained in $\mathbf{C}_\bullet(K;-)$ gives us an additive functor*

$$\mathbf{C}_\bullet(K;-) : \mathrm{Cosh}(K) \to \mathbf{Ch}(\mathbf{Vect}_\mathbb{F})$$

and this functor is exact.

Proof. To show it is a functor, we need to show it preserves composition and identities. Given an identity morphism $\mathrm{id}_\mathcal{F}$ on $\mathcal{F}$, by definition we have

$$\mathbf{C}_k(K;\mathrm{id}_\mathcal{F}) = \begin{pmatrix} \mathrm{id}_{\mathcal{F}(\sigma_1)} & & \\ & \ddots & \\ & & \mathrm{id}_{\mathcal{F}(\sigma_n)} \end{pmatrix} = \mathrm{id}_{\mathbf{C}_k(K;\mathcal{F})}$$

Given $\eta : \mathcal{F} \to \mathcal{G}$ and $\epsilon : \mathcal{G} \to \mathcal{H}$, again by definition

$$\begin{aligned} \mathbf{C}_k(K;\epsilon\circ\eta) &= \begin{pmatrix} (\epsilon\circ\eta)_{\sigma_1} & & \\ & \ddots & \\ & & (\epsilon\circ\eta)_{\sigma_n} \end{pmatrix} \\ &= \begin{pmatrix} \epsilon_{\sigma_1}\circ\eta_{\sigma_1} & & \\ & \ddots & \\ & & \epsilon_{\sigma_n}\circ\eta_{\sigma_n} \end{pmatrix} \\ &= \begin{pmatrix} \epsilon_{\sigma_1} & & \\ & \ddots & \\ & & \epsilon_{\sigma_n} \end{pmatrix} \circ \begin{pmatrix} \eta_{\sigma_1} & & \\ & \ddots & \\ & & \eta_{\sigma_n} \end{pmatrix} \\ &= \mathbf{C}_k(K;\epsilon)\circ\mathbf{C}_k(K;\eta) \end{aligned}$$

The second equality holds by the vertical composition of natural transformations, and the third equality holds by block matrix multiplication. The above shows that $\mathbf{C}_\bullet(K;-)$ is a well-defined functor.

For additivity, we need to show that for any two $\eta,\epsilon : \mathcal{F} \to \mathcal{G}$,

$$\mathbf{C}_k(K;\epsilon+\eta) = \mathbf{C}_k(K;\epsilon) + \mathbf{C}_k(K;\eta)$$

By definition, for any $\sigma \in K$ such that $\dim\sigma = k$ and any $v \in \mathcal{F}(\sigma)$, we have

$$\begin{aligned} \mathbf{C}_k(K;\epsilon+\eta)(v) &= (\epsilon+\eta)_\sigma(v) \\ &= \epsilon_\sigma(v) + \eta_\sigma(v) \\ &= \mathbf{C}_k(K;\epsilon)(v) + \mathbf{C}_k(K;\eta)(v) \end{aligned}$$

$$= (\mathbf{C}_k(K;\epsilon) + \mathbf{C}_k(K;\eta))(v)$$

The first and third equality holds by the definition of the induced chain map; the second and forth equality holds by the additive structure in $\mathrm{Cosh}(K)$ and $\mathbf{Ch}(\mathbf{Vect}_{\mathbb{F}})$, respectively. This then proves that $\mathbf{C}_\bullet(K;-)$ is an additive functor.

As for exactness, suppose we have an exact sequence in $\mathrm{Cosh}(K)$,

$$\mathcal{F} \xrightarrow{\eta} \mathcal{G} \xrightarrow{\epsilon} \mathcal{H}$$

Then by Proposition 1.7 for any $\sigma \in K$ the following sequence is exact,

$$\mathcal{F}(\sigma) \xrightarrow{\eta_\sigma} \mathcal{G}(\sigma) \xrightarrow{\epsilon_\sigma} \mathcal{H}(\sigma)$$

and so does the following direct sum of exact sequences,

$$\bigoplus_{\dim \sigma = k} \left(\ \mathcal{F}(\sigma) \xrightarrow{\eta_\sigma} \mathcal{G}(\sigma) \xrightarrow{\epsilon_\sigma} \mathcal{H}(\sigma) \ \right)$$

which by definition is the induced sequence of chain complexes (remember that the category of chain complexes is a full subcategory of a functor category, thus direct sums are computed point-wise),

$$\mathbf{C}_k(K;\mathcal{F}) \xrightarrow{\mathbf{C}_k(K;\eta)} \mathbf{C}_k(K;\mathcal{G}) \xrightarrow{\mathbf{C}_k(K;\epsilon)} \mathbf{C}_k(K;\mathcal{H})$$

Hence, the induced sequence of chain complexes is also exact, which proves the exactness of the functor $\mathbf{C}_k(K;-)$. □

Exactness of this functor in some sense assures us that the information contained in the chain complex $\mathbf{C}_\bullet(K;\mathcal{F})$, especially its homology groups, faithfully represent the information of the cosheaf $\mathcal{F}$ on K.

3. Filtration of Cosheaves and Persistent Homology

We've successfully established a functor from cosheaves on K to chain complexes, and we've further shown that this functor is exact. In this section, we will use this functor to study the filtration of a cosheaf, and define its persistent homology.

Definition 3.1. *A filtration of a cosheaf $\mathcal{F}$ on K constitutes a family of injective morphisms in* $\mathrm{Cosh}(K)$

$$\eta_i : \mathcal{F}_i \hookrightarrow \mathcal{F}_{i+1},\ 1 \le i \le n-1$$

such that $\mathcal{F}_n = \mathcal{F}$. For every $1 \le i < j \le n$ we write $\eta_{i\to j}$ as the following composite,

$$\eta_{i\to j} := \eta_{j-1} \circ \cdots \circ \eta_i$$

and $\eta_{i\to i}$ is simply defined as the identity $\mathrm{id}_{\mathcal{F}_i}$.

We will write a filtration of $\mathcal{F}$ as the following sequence in $\mathrm{Cosh}(K)$,

$$\mathcal{F}_1 \hookrightarrow \mathcal{F}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_n$$

Given a filtration, we can apply the functor $\mathbf{C}_\bullet(K;-)$ and obtain a sequence of chain complexes

$$\mathbf{C}_\bullet(K;\mathcal{F}_1) \hookrightarrow \mathbf{C}_\bullet(K;\mathcal{F}_2) \hookrightarrow \cdots \hookrightarrow \mathbf{C}_\bullet(K;\mathcal{F}_n)$$

By Proposition 2.9 the functor $\mathbf{C}_\bullet(K;-)$ is exact. In particular, it should preserves injectives, and thus the chain maps between these chain complexes would also be injective.

The persistent homology of a filtration of cosheaves is defined as follows:

Definition 3.2. *Given a filtration of the cosheaf $\mathcal{F}$ in* $\mathrm{Cosh}(K)$

$$\mathcal{F}_1 \xhookrightarrow{\eta_1} \mathcal{F}_2 \xhookrightarrow{\eta_2} \cdots \xhookrightarrow{\eta_{n-1}} \mathcal{F}_n$$

The k-th persistent homology group for any pair $i \le j$ of this filtration is defined as follows

$$\mathbf{PH}_{i\to j,k}\mathcal{F}_\bullet := \mathbf{PH}_{i\to j}\mathbf{H}_k(K;\mathcal{F}_\bullet)$$

where $\mathbf{H}_k(L;\mathcal{F}_\bullet)$ is the following persistent module

$$\mathbf{H}_k(K;\mathcal{F}_1) \xhookrightarrow{\mathbf{H}_k(K;\eta_1)} \mathbf{H}_k(K;\mathcal{F}_2) \xhookrightarrow{\mathbf{H}_k(K;\eta_2)} \cdots \xhookrightarrow{\mathbf{H}_k(K;\eta_{n-1})} \mathbf{H}_k(K;\mathcal{F}_n)$$

This persistent module is obtained by functoriality of the k-th homology group.

The main goal of this essay is to simplify the computation of the k-th persistent homology group of a filtration of a cosheaf using Morse theory. The crucial result that the whole process relies on is the following proposition:

Proposition 3.3. *Given two additive functors*

$$F, G : \mathcal{B} \to \mathbf{Ch}(\mathcal{A})$$

where $\mathcal{A}, \mathcal{B}$ are abelian categories, if there exist natural transformations

$$\psi : F \to G,\ \phi : G \to F$$

such that they are naturally weak equivalent, i.e. for any $x \in \mathcal{B}$, there exist two homotopies,

$$\theta_x : \mathrm{id}_{Fx} \simeq (\phi \circ \psi)_x,\ \gamma_x : \mathrm{id}_{Gx} \simeq (\phi \circ \psi)_x$$

where for any $f : x \to y$ in $\mathcal{B}$ the following diagrammes commute for any level k,

$$\begin{array}{ccc} (Fx)_k & \xrightarrow{(Ff)_k} & (Fy)_k \\ \theta_{x,k}\downarrow & & \downarrow\theta_{y,k} \\ (Fx)_{k+1} & \xrightarrow[(Ff)_{k+1}]{} & (Fy)_{k+1} \end{array} \qquad \begin{array}{ccc} (Gx)_k & \xrightarrow{(Gf)_k} & (Gy)_k \\ \gamma_{x,k}\downarrow & & \downarrow\gamma_{y,k} \\ (Gx)_{k+1} & \xrightarrow[(Gf)_{k+1}]{} & (Gy)_{k+1} \end{array}$$

Then for any sequence in $\mathcal{B}$,

$$x_1 \to x_2 \to \cdots \to x_n$$

the persistent homology of the two chain sequences are isomorphic

$$\mathbf{PH}_{i\to j,k}Fx_\bullet \cong \mathbf{PH}_{i\to j,k}Gx_\bullet$$

Proof. Give any sequence in $\mathcal{B}$

$$x_1 \to x_2 \to \cdots \to x_n$$

The two functors induces two sequences of chain complexes, and the natural transformation provides the following commuting diagramme,

$$\begin{array}{ccccccccc} Fx_1 & \longrightarrow & Fx_2 & \longrightarrow & \cdots & \longrightarrow & Fx_{n-1} & \longrightarrow & Fx_n \\ \psi_{x_1}\downarrow & & \psi_{x_2}\downarrow & & & & \psi_{x_{n-1}}\downarrow & & \psi_{x_n}\downarrow \\ Gx_1 & \longrightarrow & Gx_2 & \longrightarrow & \cdots & \longrightarrow & Gx_{n-1} & \longrightarrow & Gx_n \end{array}$$

By assumption, the natural transformation ϕ gives out quasi-isomorphisms ϕ_{x_i} for each x_i, and in addition the chain homotopies between these quasi-isomorphisms are natural. Hence, it restricts to the following commuting diagramme between homology groups at any level k,

$$\begin{array}{ccccccccc} \mathbf{H}_k Fx_1 & \longrightarrow & \mathbf{H}_k Fx_2 & \longrightarrow & \cdots & \longrightarrow & \mathbf{H}_k Fx_{n-1} & \longrightarrow & \mathbf{H}_k Fx_n \\ \cong\downarrow & & \cong\downarrow & & & & \cong\downarrow & & \cong\downarrow \\ \mathbf{H}_k Gx_1 & \longrightarrow & \mathbf{H}_k Gx_2 & \longrightarrow & \cdots & \longrightarrow & \mathbf{H}_k Gx_{n-1} & \longrightarrow & \mathbf{H}_k Gx_n \end{array}$$

Then it implies the two sequences have the same persistent homology groups. □

In the next section, we will explicitly describe an alternative functor using discrete Morse theory, which we intend to show its equivalence with the functor $\mathbf{C}_\bullet(K; -)$ we've already defined in the sense of Proposition 3.3.

4. Functor of Associated Morse Complex

In the light of discussions at the end of last section, we want to describe another functor from the category of cosheaves to the category of chain complexes, which is equivalent to the functor $\mathbf{C}_\bullet(K; -)$ we've already constructed when computing homologies and persistent homologies, but lands in smaller and more controllable chain complexes that are easier to compute. The method we use is to introduce an acyclic partial matching Σ.

Definition 4.1. *A partial matching Σ of K is a set of pairs $(\sigma \lhd \tau)$ of simplices in K such that no two pairs have a common simplex. A path ρ of Σ is a zigzag sequence of simplices of the following form*

$$\rho = (\sigma_1 \lhd \tau_1 \rhd \sigma_2 \lhd \tau_2 \rhd \cdots \rhd \sigma_m \lhd \tau_m)$$

where each $(\sigma_i \lhd \tau_i) \in \Sigma$. We write σ_ρ and τ_ρ for the start and end simplex of a path ρ respectively. ρ is gradient if either $m = 1$ or σ_1 is not a face of τ_m. A partial matching Σ is acyclic if all its paths are gradient. A Σ-critical simplex $\alpha \in K$ is a simplex that do not appear in any pairs of Σ.

In the following texts we will always assume that Σ is an acyclic partial matching on K. We will use the data of Σ to construct a smaller chain complex associated to a cosheaf. To be more precise, given an acyclic partial matching Σ, what we will construct is a functor

$$\mathbf{C}^\Sigma_\bullet(K; -) : \mathrm{Cosh}^\Sigma(K) \to \mathbf{Ch}(\mathbf{Vect}_\mathbb{F})$$

where $\mathrm{Cosh}^{\Sigma}(K)$, the category of Σ-compatible cosheaves, is a full subcategory of $\mathrm{Cosh}(K)$ which is also abelian. The construction of such a functor is the main goal of this section. We can also restrict the domain of the functor $\mathbf{C}_{\bullet}(K;-)$ to $\mathrm{Cosh}^{\Sigma}(K)$, so that the two functors will now have the same domain and codomain. Their equivalence will be proved in the next section.

We first need to say what the category $\mathrm{Cosh}^{\Sigma}(K)$ is when given an acyclic partial matching Σ.

Definition 4.2. *A cosheaf $\mathcal{F}$ on K is Σ-compatible if for each $(\sigma \lhd \tau) \in \Sigma$, the restriction map $\mathcal{F}(\sigma \leq \tau)$ is an isomorphism.*

Definition 4.3. *The category $\mathrm{Cosh}^{\Sigma}(K)$ of Σ-compatible cosheaves is the full subcategory of $\mathrm{Cosh}(K)$ where objects are Σ-compatible cosheaves.*

Proposition 4.4. *The category $\mathrm{Cosh}^{\Sigma}(K)$ is also abelian.*

Proof. Since $\mathrm{Cosh}^{\Sigma}(K)$ is a full subcategory of the abelian category $\mathrm{Cosh}(K)$, it suffices to show that it is closed under zero object, direct sums and kernels and cokernels. The zero cosheaf $\underline{0}$ is obviously Σ-compatible, because all its restriction maps are isomorphisms. For any Σ-compatible cosheaf $\mathcal{F}$ and $\mathcal{G}$, as already discussed in Section 1, their direct sum in $\mathrm{Cosh}(K)$ is the point-wise direct sum. In particular, for any $(\sigma \lhd \tau) \in \Sigma$ we have

$$(\mathcal{F} \oplus \mathcal{G})(\sigma \leq \tau) = \mathcal{F}(\sigma \leq \tau) \oplus \mathcal{G}(\sigma \leq \tau)$$

If both $\mathcal{F}(\sigma \leq \tau)$ and $\mathcal{G}(\sigma \leq \tau)$ are isomorphisms, then so does their direct sum. It follows that $\mathcal{F} \oplus \mathcal{G}$ is also Σ-compatible. Given any morphism $\eta : \mathcal{F} \to \mathcal{G}$ between Σ-compatible cosheaves, by Lemma 1.2 the kernel of η is computed point-wise as the following commuting diagramme shows,

$$\begin{array}{ccccc} \ker \eta_\sigma & \longrightarrow & \mathcal{F}(\sigma) & \xrightarrow{\eta_\sigma} & \mathcal{G}(\sigma) \\ {\scriptstyle (\ker \eta)(\sigma \leq \tau)}\uparrow & & \cong\uparrow & & \cong\uparrow \\ \ker \eta_\tau & \longrightarrow & \mathcal{F}(\tau) & \xrightarrow{\eta_\tau} & \mathcal{G}(\tau) \end{array}$$

Now by the universal property of kernels, it follows the induced canonical map

$$(\ker \eta)(\sigma \leq \tau) : \ker \eta_\tau \to \ker \eta_\sigma$$

must also be an isomorphism. The case for the cokernel is similar. This then shows that both $\ker \eta$ and $\operatorname{coker} \eta$ are Σ-compatible. Hence, the category $\mathrm{Cosh}^{\Sigma}(K)$ is also abelian. □

What we will consider in the following texts are filtrations in $\mathrm{Cosh}^{\Sigma}(K)$, which requires every cosheaf appearing in the sequence is Σ-compatible. Proposition 4.4 then makes sure that what we have defined and discussed so far, especially of the definition of persistent homology groups, also applies to the category $\mathrm{Cosh}^{\Sigma}(K)$.

We can now proceed to define the data of the functor $\mathbf{C}^{\Sigma}_{\bullet}(K;-)$. Let C^k_Σ be the following set

$$C^k_\Sigma := \{\, \alpha \in K \mid \dim \alpha = k \ \&\ \alpha \text{ is } \Sigma\text{-critical} \,\}$$

Then the chain groups of the Morse complex and the boundary maps associated to a Σ-compatible cosheaf $\mathcal{F}$ is defined as follows:

Definition 4.5. *For $\mathcal{F} \in \mathrm{Cosh}^{\Sigma}(K)$, the k-th chain group of its Morse complex is*

$$\mathbf{C}_k^{\Sigma}(K;\mathcal{F}) := \bigoplus_{\alpha \in C_{\Sigma}^k} \mathcal{F}(\alpha)$$

As you can see, $\mathbf{C}_k^{\Sigma}(K;\mathcal{F})$ is indexed by k-simplices α that are critical. If Σ is not empty, then we would result in smaller chain groups than the original complex $\mathbf{C}_{\bullet}(K;\mathcal{F})$.

To define the boundary maps, we need some priori notions.

Definition 4.6. *Given a Σ-path ρ*

$$\rho = (\sigma_1 \lhd \tau_1 \rhd \cdots \rhd \sigma_m \lhd \tau_m)$$

for any $\mathcal{F} \in \mathrm{Cosh}^{\Sigma}(K)$, the $\mathcal{F}$-weight of ρ is a map $w_{\mathcal{F}}(\rho) : \mathcal{F}(\sigma_1) \to \mathcal{F}(\tau_m)$ defined as follows,

$$w_{\mathcal{F}}(\rho) = (-)^m \cdot \mathcal{F}_{\tau_m,\sigma_m}^{-1} \circ \mathcal{F}_{\tau_{m-1},\sigma_m} \circ \cdots \circ \mathcal{F}_{\tau_1,\sigma_2} \circ \mathcal{F}_{\tau_1,\sigma_1}^{-1}$$

This is well-defined, because we require a Σ-compatible cosheaf $\mathcal{F}$ to have invertible maps $\mathcal{F}(\sigma \leq \tau)$ for any $(\sigma \lhd \tau) \in \Sigma$. By Definition 2.2, the map $\mathcal{F}_{\tau,\sigma}$ is the map $\mathcal{F}(\sigma \leq \tau)$ multiplied by a scalar, hence it is also invertible.

We can now define the boundary maps of the Morse complex:

Definition 4.7. *Given $\mathcal{F} \in \mathrm{Cosh}^{\Sigma}(K)$, the k-th boundary map of the Morse complex*

$$d_k^{\Sigma,\mathcal{F}} : \mathbf{C}_k^{\Sigma}(K;\mathcal{F}) \to \mathbf{C}_{k-1}^{\Sigma}(K;\mathcal{F})$$

is defined block-wise; given any $\alpha \in C_{\Sigma}^k$ and any $\omega \in C_{\Sigma}^{k-1}$, the α,ω-component of the boundary map $d_k^{\Sigma,\mathcal{F}}|_{\alpha,\omega} : \mathcal{F}(\alpha) \to \mathcal{F}(\omega)$ is given by

$$d_k^{\Sigma,\mathcal{F}}|_{\alpha,\omega} := \mathcal{F}_{\alpha,\omega} + \sum_{\rho} \mathcal{F}_{\tau_\rho,\omega} \circ w_{\mathcal{F}}(\rho) \circ \mathcal{F}_{\alpha,\sigma_\rho}$$

The proof strategy of showing $(\mathbf{C}_{\bullet}^{\Sigma}(K;\mathcal{F}), d_{\bullet}^{\Sigma,\mathcal{F}})$ forms a chain complex is by induction on the cardinality of Σ. More precisely, it is shown in [3] that each time you reduce a single pair $(\sigma \lhd \tau) \in \Sigma$ properly, you would result in the same Morse data. Hence, in future proofs, it suffices to assume there is only one pair in Σ, and the rest follows inductively by arguments in [3].

Proposition 4.8. *For any $\mathcal{F} \in \mathrm{Cosh}^{\Sigma}(K)$ and any $k \in \mathbb{N}$, we have*

$$d_k^{\Sigma,\mathcal{F}} \circ d_{k+1}^{\Sigma,\mathcal{F}} = 0$$

Proof. Due to the above discussion, we may assume there is only one pair $(\sigma \lhd \tau)$ in Σ. The only non-trivial cases we need to verify are for some α, ω such that either $\alpha \rhd \sigma$ or $\omega \lhd \tau$. If $\alpha \rhd \sigma$ and $\dim \alpha = k+1$, then by definition we have

$$\left(d_k^{\Sigma,\mathcal{F}} \circ d_{k+1}^{\Sigma,\mathcal{F}}\right)|_{\alpha,\omega} = \sum_{\delta \in C_{\Sigma}^k} \mathcal{F}_{\delta,\omega} \circ \mathcal{F}_{\alpha,\delta} + \sum_{\delta \in C_{\Sigma}^k} \mathcal{F}_{\delta,\omega} \circ \mathcal{F}_{\tau,\delta} \circ w_{\mathcal{F}}(\sigma \lhd \tau) \circ \mathcal{F}_{\alpha,\sigma}$$

$$= -\mathcal{F}_{\sigma,\omega} \circ \mathcal{F}_{\alpha,\sigma} - \mathcal{F}_{\sigma,\omega} \circ \mathcal{F}_{\tau,\sigma} \circ w_{\mathcal{F}}(\sigma \lhd \tau) \circ \mathcal{F}_{\alpha,\sigma}$$
$$= -\mathcal{F}_{\sigma,\omega} \circ \mathcal{F}_{\alpha,\sigma} + \mathcal{F}_{\sigma,\omega} \circ \mathcal{F}_{\tau,\sigma} \circ \mathcal{F}_{\tau,\sigma}^{-1} \circ \mathcal{F}_{\alpha,\sigma} = 0$$

The first equality holds by definition; the second equality holds by Lemma 2.3: the only k-simplex not in C_Σ^k is σ, hence if summing over all k-simplices results in zero then summing over all simplices in C_Σ^k would be the negative of summing over σ. The rest is straight forward calculation. The case where $\omega \lhd \tau$ is similar. □

Corollary 4.9. *For any cosheaf $\mathcal{F} \in \mathrm{Cosh}^\Sigma(K)$, the associated Morse complex $\mathbf{C}_\bullet^\Sigma(K;\mathcal{F})$ is indeed a chain complex.*

This completes the definition of how $\mathbf{C}_\bullet^\Sigma(K;-)$ acts on objects. We then proceed to describe how such construction acts on morphisms. Like in the case of $\mathbf{C}_\bullet(K;-)$, everything follows naturally.

Definition 4.10. *Given any $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Cosh}^\Sigma(K)$, the k-th chain map*

$$\mathbf{C}_k^\Sigma(K;\eta) : \mathbf{C}_k^\Sigma(K;\mathcal{F}) \to \mathbf{C}_k^\Sigma(K;\mathcal{G})$$

is given by block-diagonal maps

$$\mathbf{C}_k^\Sigma(K;\eta) := \bigoplus_{\sigma \in C_\Sigma^k} \eta_\sigma : \bigoplus_{\sigma \in C_\Sigma^k} \mathcal{F}(\sigma) \to \bigoplus_{\sigma \in C_\Sigma^k} \mathcal{G}(\sigma)$$

To prove such construction is natural, we first observe that due to naturality of $\eta : \mathcal{F} \to \mathcal{G}$, it "commutes" with all our constructions using the cosheaf data so far:

Lemma 4.11. *For any $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Cosh}^\Sigma(K)$, for any simplices $\sigma, \tau \in K$ we have*

$$\mathcal{G}_{\tau,\sigma} \circ \eta_\tau = \eta_\sigma \circ \mathcal{F}_{\tau,\sigma}$$

For any Σ-path ρ, we also have

$$w_{\mathcal{G}}(\rho) \circ \eta_{\sigma_\rho} = \eta_{\tau_\rho} \circ w_{\mathcal{F}}(\rho)$$

Proof. By naturality of η, given any $\sigma \leq \tau$ the following holds,

$$\mathcal{G}(\sigma \leq \tau) \circ \eta_\tau = \eta_\sigma \circ \mathcal{F}(\sigma \leq \tau)$$

This implies we can multiply by a scalar and obtain

$$\mathcal{G}_{\tau,\sigma} \circ \eta_\tau = \eta_\sigma \circ \mathcal{F}_{\tau,\sigma}$$

If we also have that $\mathcal{G}(\sigma \leq \tau)$ and $\mathcal{F}(\sigma \leq \tau)$ are invertible, then the above also implies

$$\eta_\tau \circ \mathcal{F}(\sigma \leq \tau)^{-1} = \mathcal{G}(\sigma \leq \tau)^{-1} \circ \eta_\sigma$$

This then implies η commutes with every term appearing in $w(\rho)$, thus we have

$$w_{\mathcal{G}}(\rho) \circ \eta_{\sigma_\rho} = \eta_{\tau_\rho} \circ w_{\mathcal{F}}(\rho)$$

Hence it completes the proof. □

Proposition 4.12. *For any $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Cosh}^{\Sigma}(K)$ and any $k \in \mathbb{N}$, the following diagramme commutes,*

$$\begin{array}{ccc} \mathbf{C}^{\Sigma}_{k}(K;\mathcal{F}) & \xrightarrow{\mathbf{C}^{\Sigma}_{k}(K;\eta)} & \mathbf{C}^{\Sigma}_{k}(K;\mathcal{G}) \\ {\scriptstyle d^{\Sigma,\mathcal{F}}_{k}}\downarrow & & \downarrow{\scriptstyle d^{\Sigma,\mathcal{G}}_{k}} \\ \mathbf{C}^{\Sigma}_{k-1}(K;\mathcal{F}) & \xrightarrow[\mathbf{C}^{\Sigma}_{k-1}(K;\eta)]{} & \mathbf{C}^{\Sigma}_{k-1}(K;\mathcal{G}) \end{array}$$

Proof. By Lemma 4.11, given $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Cosh}^{\Sigma}(K)$, it commutes with all the terms appearing in the construction of the boundary map. For any $\alpha \in C^{k}_{\Sigma}$ and any $\omega \in C^{k-1}_{\Sigma}$ we have

$$\begin{aligned} \left(\mathbf{C}^{\Sigma}_{k-1}(K;\eta) \circ d^{\Sigma,\mathcal{F}}_{k}\right)|_{\alpha,\omega} &= \eta_{\omega} \circ \left(\mathcal{F}_{\alpha,\omega} + \sum_{\rho} \mathcal{F}_{\tau_{\rho},\omega} \circ w_{\mathcal{F}}(\rho) \circ \mathcal{F}_{\alpha,\sigma_{\rho}}\right) \\ &= \eta_{\omega} \circ \mathcal{F}_{\alpha,\omega} + \sum_{\rho} \eta_{\omega} \circ \mathcal{F}_{\tau_{\rho},\omega} \circ w_{\mathcal{F}}(\rho) \circ \mathcal{F}_{\alpha,\sigma_{\rho}} \\ &= \mathcal{G}_{\alpha,\omega} \circ \eta_{\alpha} + \sum_{\rho} \mathcal{G}_{\tau_{\rho},\omega} \circ w_{\mathcal{G}}(\rho) \circ \mathcal{G}_{\alpha,\sigma_{\rho}} \circ \eta_{\alpha} \\ &= \left(\mathcal{G}_{\alpha,\omega} + \sum_{\rho} \mathcal{G}_{\tau_{\rho},\omega} \circ w_{\mathcal{G}}(\rho) \circ \mathcal{G}_{\alpha,\sigma_{\rho}}\right) \circ \eta_{\alpha} \\ &= \left(d^{\Sigma,\mathcal{G}}_{k} \circ \mathbf{C}^{\Sigma}_{k}(K;\eta)\right)|_{\alpha,\omega} \end{aligned}$$

This holds for any block, thus indeed the diagramme commutes. □

Corollary 4.13. *Any morphism $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Cosh}^{\Sigma}(K)$ naturally induces a chain map between their Morse complexes,*

$$\mathbf{C}^{\Sigma}_{\bullet}(K;\eta) : \mathbf{C}^{\Sigma}_{\bullet}(K;\mathcal{F}) \to \mathbf{C}^{\Sigma}_{\bullet}(K;\mathcal{G})$$

Proof. It's a direct consequence of Proposition 4.12. □

The proof of functoriality, additivity and exactness of $\mathbf{C}^{\Sigma}_{\bullet}(K;-)$ is essentially the same as in Proposition 2.9, which all come down to the computation of block-diagonal matrices.

Proposition 4.14. *The data contained in $\mathbf{C}^{\Sigma}_{\bullet}(K;-)$ gives an exact additive functor*

$$\mathbf{C}^{\Sigma}_{\bullet}(K;-) : \mathrm{Cosh}^{\Sigma}(K) \to \mathbf{Ch}(\mathbf{Vect}_{\mathbb{F}})$$

Proof. Essentially the same as Proposition 2.9. □

5. Establishing the Equivalence

In the previous sections, we've already constructed our two main functors that associates each Σ-compatible cosheaf to a chain complex,

$$\mathbf{C}_{\bullet}(K;-), \mathbf{C}^{\Sigma}_{\bullet}(K;-) : \mathrm{Cosh}^{\Sigma}(K) \to \mathbf{Ch}(\mathbf{Vect}_{\mathbb{F}})$$

In this section, our aim is to construct two natural transformations and natural homotopies in the sense of Proposition 3.3, which then allows us to conclude that the two functors induce the same persistent homology groups on any sequence in $\mathrm{Cosh}^{\Sigma}(K)$.

The work left to us is essentially easy, since the construction of two natural transformations and a homotopy are essentially already given in [3]. Though it constructs and proves the case for sheaves and cohomology, all of those can be directly translated into our case by formally reversing the arrows. Also, as already mentioned in previous sections as well, by induction it suffices to show how the equivalence is built assuming Σ only contains one pair of simplices. Thus, from now we assume $\Sigma = \{(\sigma \lhd \tau)\}$. The first task is to construct two natural transformations

$$\psi_\bullet : \mathbf{C}_\bullet(K;-) \to \mathbf{C}^{\Sigma}_\bullet(K;-),\ \phi_\bullet : \mathbf{C}^{\Sigma}_\bullet(K;-) \to \mathbf{C}_\bullet(K;-)$$

We define them block-wise: for any α, ω such that $\alpha \in K, \dim\alpha = k$ and $\omega \in C^k_\Sigma$, we have

$$\psi^{\mathcal{F}}_k|_{\alpha,\omega} := \begin{cases} \mathcal{F}_{\tau,\omega} \circ w_{\mathcal{F}}(\sigma \lhd \tau) & \alpha = \sigma \\ \mathrm{id}_{\mathcal{F}(\alpha)} & \alpha = \omega \\ 0 & \text{otherwise} \end{cases}$$

and for ϕ we define

$$\phi^{\mathcal{F}}_k|_{\omega,\alpha} := \begin{cases} w_{\mathcal{F}}(\sigma \lhd \tau) \circ \mathcal{F}_{\omega,\sigma} & \alpha = \tau \\ \mathrm{id}_{\mathcal{F}(\alpha)} & \alpha = \omega \\ 0 & \text{otherwise} \end{cases}$$

Proposition 5.1. *For any cosheaf $\mathcal{F}$, $\psi^{\mathcal{F}}_\bullet$ and $\phi^{\mathcal{F}}_\bullet$ are well-defined chain maps.*

Proof. For any $\alpha \in K$ that $\dim\alpha = k$ and any $\omega \in C^{k-1}_\Sigma$: (1) if $\alpha \in C^k_\Sigma$, then by definition we would have

$$(d^{\Sigma,\mathcal{F}}_k \circ \psi^{\mathcal{F}}_k)|_{\alpha,\omega} = d^{\Sigma,\mathcal{F}}_k|_{\alpha,\omega} = \mathcal{F}_{\alpha,\omega} + \mathcal{F}_{\tau,\omega} \circ w_{\mathcal{F}}(\sigma \lhd \tau) \circ \mathcal{F}_{\alpha,\sigma}$$

On the other hand,

$$(\psi^{\mathcal{F}}_{k-1} \circ d^{\mathcal{F}}_k)|_{\alpha,\omega} = \psi^{\mathcal{F}}_{k-1} \circ \left(\sum_{\delta \lhd \alpha} \mathcal{F}_{\alpha,\delta} \right)$$

By definition, $\mathcal{F}_{\alpha,\omega} \neq 0$ iff $\omega \lhd \alpha$, and so is $\mathcal{F}_{\alpha,\sigma}$. It follows that expanding the summation above we would also have

$$(\psi^{\mathcal{F}}_{k-1} \circ d^{\mathcal{F}}_k)|_{\alpha,\omega} = \mathcal{F}_{\alpha,\omega} + \mathcal{F}_{\tau,\omega} \circ w_{\mathcal{F}}(\sigma \lhd \tau) \circ \mathcal{F}_{\alpha,\sigma}$$

This shows that indeed we have

$$(d^{\Sigma,\mathcal{F}}_k \circ \psi^{\mathcal{F}}_k)|_{\alpha,\omega} = (\psi^{\mathcal{F}}_{k-1} \circ d^{\mathcal{F}}_k)|_{\alpha,\omega}$$

which would then imply ψ is a well-defined chain map. The argument of ϕ being a chain map is essentially similar. □

We temporarily postpone the proof of naturality of $\psi_\bullet$ and $\phi_\bullet$. Let me first describte how the homotopies are built, and then prove the naturality of all of these maps together. First, we observe that

$$\psi_\bullet \circ \phi_\bullet = \mathrm{id}$$

Given any $\omega \in C^k_\Sigma$, $\phi^{\mathcal{F}}_k$ is the following map

$$\phi^{\mathcal{F}}_k|_{\omega,-} = \mathrm{id}_{\mathcal{F}(\omega)} \oplus w_{\mathcal{F}}(\sigma \lhd \tau) \circ \mathcal{F}_{\omega,\sigma}$$

If we then apply $\psi^{\mathcal{F}}_k$, the second summand above would be mapped to zero, because it lands in τ, and by our definition of $\psi^{\mathcal{F}}_k$ when $\alpha = \tau$ it is mapped to zero. Hence, the only thing left is the identity map, thus indeed we have

$$\psi^{\mathcal{F}}_k \circ \phi^{\mathcal{F}}_k = \mathrm{id}_{\mathbf{C}^{\Sigma}_k(K;\mathcal{F})}$$

It follows that we only need to construct a homotopy of the other direction. A homotopy from identity to $\phi^{\mathcal{F}}_\bullet \circ \psi^{\mathcal{F}}_\bullet$ is a family of maps $\theta^{\mathcal{F}}_\bullet$,

$$\theta^{\mathcal{F}}_k : \mathbf{C}_k(K;\mathcal{F}) \to \mathbf{C}_{k+1}(K;\mathcal{F})$$

for each level k. It is defined block-wise for any $\dim \alpha = k$ and any $\dim \omega = k+1$ as follows

$$\theta^{\mathcal{F}}_k|_{\alpha,\omega} := \begin{cases} \mathcal{F}^{-1}_{\tau,\sigma} & \alpha = \sigma \;\&\; \omega = \tau \\ 0 & \text{otherwise} \end{cases}$$

Proposition 5.2. *$\theta^{\mathcal{F}}_\bullet$ constitutes a chain homotopy from identity to $\phi^{\mathcal{F}}_\bullet \circ \psi^{\mathcal{F}}_\bullet$ for any $\mathcal{F} \in \mathrm{Cosh}^{\Sigma}(K)$.*

Proof. The only non-trivial case is when $\alpha = \sigma$. By definition,

$$\begin{aligned}(\phi^{\mathcal{F}}_k \circ \psi^{\mathcal{F}}_k)|_{\alpha,-} &= \phi^{\mathcal{F}}_k \circ \left(\oplus_{\delta \lhd \tau, \delta \neq \sigma} \mathcal{F}_{\tau,\delta} \circ w_{\mathcal{F}}(\sigma \lhd \tau)\right) \\ &= - \oplus_{\delta \lhd \tau, \delta \neq \sigma} \mathcal{F}_{\tau,\delta} \circ \mathcal{F}^{-1}_{\tau,\sigma}\end{aligned}$$

The homotopy computes as follows

$$\begin{aligned}(\theta^{\mathcal{F}}_{k-1} \circ d^{\mathcal{F}}_k + d^{\mathcal{F}}_{k+1} \circ \theta^{\mathcal{F}}_k)|_{\alpha,-} &= d^{\mathcal{F}}_{k+1} \circ \mathcal{F}^{-1}_{\tau,\sigma} \\ &= \bigoplus_{\delta \lhd \tau} \mathcal{F}_{\tau,\delta} \circ \mathcal{F}^{-1}_{\tau,\sigma}\end{aligned}$$

The first equality holds because when $\alpha = \sigma$, we obviously have

$$(\theta^{\mathcal{F}}_{k-1} \circ d^{\mathcal{F}}_k)|_{\alpha,-} = 0$$

since the faces of σ are definitely not σ and θ maps them to zero. Now as for the identity map on $\alpha = \sigma$, we have

$$\mathrm{id}_{\mathcal{F}(\alpha)} = \mathcal{F}_{\tau,\sigma} \circ \mathcal{F}^{-1}_{\tau,\sigma}$$

Then indeed it follows that

$$\begin{aligned}(\mathrm{id}_{\mathbf{C}_k(K;\mathcal{F})} - (\phi^{\mathcal{F}}_k \circ \psi^{\mathcal{F}}_k))|_{\alpha,-} &= \mathcal{F}_{\tau,\sigma} \circ \mathcal{F}^{-1}_{\tau,\sigma} + \oplus_{\delta \lhd \tau, \delta \neq \sigma} \mathcal{F}_{\tau,\delta} \circ \mathcal{F}^{-1}_{\tau,\delta} \\ &= \bigoplus_{\delta \lhd \tau} \mathcal{F}_{\tau,\delta} \circ \mathcal{F}^{-1}_{\tau,\sigma} \\ &= (\theta^{\mathcal{F}}_{k-1} \circ d^{\mathcal{F}}_k + d^{\mathcal{F}}_{k+1} \circ \theta^{\mathcal{F}}_k)|_{\alpha,-}\end{aligned}$$

This shows that $\theta^{\mathcal{F}}_\bullet$ is indeed a homotopy between identity and $\phi^{\mathcal{F}}_\bullet \circ \psi^{\mathcal{F}}_\bullet$. □

Now according to Proposition 3.3, the final step is to verify the naturality of the constructions $\phi^{\mathcal{F}}_\bullet$, $\psi^{\mathcal{F}}_\bullet$ and $\theta^{\mathcal{F}}_\bullet$:

Proposition 5.3. *The constructions $\phi_\bullet^{\mathcal{F}}, \psi_\bullet^{\mathcal{F}}$ and $\theta_\bullet^{\mathcal{F}}$ are all natural in $\mathcal{F}$ in the sense of Proposition 3.3. Explicitly, $\phi_\bullet, \psi_\bullet$ are natural transformations*

$$\psi_\bullet : \mathbf{C}_\bullet(K; -) \rightleftarrows \mathbf{C}_\bullet^\Sigma(K; -) : \phi_\bullet$$

and $\theta_\bullet$ is a natural homotopy.

Proof. The naturality proof again essentially relies on the naturality of any morphism $\eta : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Cosh}^\Sigma(K)$. In the definition of $\psi_\bullet^{\mathcal{F}}, \phi_\bullet^{\mathcal{F}}$ and $\theta_\bullet^{\mathcal{F}}$, all we use are 0, identities, and morphisms of the form $\mathcal{F}_{\alpha,\omega}$ or $w_{\mathcal{F}}(\sigma \lhd \tau)$ for some α, ω. Now 0 maps and identity maps obviously commutes with any morphism; from Lemma 4.11 we also know that η commutes with the rest of the maps. This then suffices to show that the constructions $\phi_\bullet, \psi_\bullet, \theta_\bullet$ are all natural. □

Corollary 5.4. *For any sequence in* $\mathrm{Cosh}^\Sigma(K)$*,*

$$\mathcal{F}_1 \hookrightarrow \mathcal{F}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_n$$

The two functors $\mathbf{C}_\bullet(K; -)$ and $\mathbf{C}_\bullet^\Sigma(K; -)$ computes the same persistent homology.

Proof. By the joint work of Proposition 3.3 and Proposition 5.2. □

We have now completed the main goal of this essay.

6. Conclusion and Future Work

In this essay, we have first introduced the functor

$$\mathbf{C}_\bullet(K; -) : \mathrm{Cosh}(K) \to \mathbf{Ch}(\mathbf{Vect}_{\mathbb{F}})$$

which describes how to do cosheaf homology theory on a simplicial complex K. We've shown its exactness, and persistent homology naturally arises when we consider the canonically induced functor

$$\mathbf{Ch}(\mathbf{C}_\bullet(K; -)) : \mathbf{Ch}(\mathrm{Cosh}(K)) \to \mathbf{Ch}(\mathbf{Ch}(\mathbf{Vect}_{\mathbb{F}}))$$

The main goal and result of this essay is to simplify the computation of such functors by discrete Morse theory. It proceeds by introducing acyclic partial matchings Σ and Σ-compatible cosheaves, which restricts our original category $\mathrm{Cosh}(K)$ of arbitrary cosheaves to a smaller one $\mathrm{Cosh}^\Sigma(K)$. We've proved that $\mathrm{Cosh}^\Sigma(K)$ is a full subcategory of $\mathrm{Cosh}(K)$ which is also abelian. Essentially, we are studying the computation of persistent homology of filtrations in $\mathrm{Cosh}^\Sigma(K)$, which are Σ-compatible filtration $\mathcal{F}_\bullet$ of a cosheaf where each $\mathcal{F}_i$ is Σ-compatible.

The domain of the original functor $\mathbf{C}_\bullet(K; -)$ can be restricted to $\mathrm{Cosh}^\Sigma(K)$, and the simplification of computation is given by introducing another functor lands in smaller chain complexes

$$\mathbf{C}_\bullet^\Sigma(K; -) : \mathrm{Cosh}^\Sigma(K) \to \mathbf{Ch}(\mathbf{Vect}_{\mathbb{F}})$$

The main result is then the two functors computes the same persistent homology groups.

However, there is a main drawback of the current framework. The requirement of Σ-compatible cosheaf is quite severe. It needs for every pair $(\sigma \lhd \tau) \in \Sigma$, the cosheaf assigns an isomorphism $\mathcal{F}(\sigma \leq \tau)$. If Σ is too large, it could make our category $\mathrm{Cosh}^{\Sigma}(K)$ substantially smaller than the original one $\mathrm{Cosh}(K)$ (though for the purpose of classical homology and cohomology it suffices, since the constant sheaf $\underline{\mathbb{F}}$ would always lie in $\mathrm{Cosh}^{\Sigma}(K)$).

The reason we require this is that we need the inverses of such maps in order to construct the weight map $w_{\mathcal{F}}(\rho)$ associated to a Σ-path ρ. Going alone the path we want maps from $\mathcal{F}(\sigma)$ to $\mathcal{F}(\tau)$ for any pair $(\sigma \triangleleft \tau) \in \Sigma$, which reverses the natural direction the cosheaf data provides. The weight map $w_{\mathcal{F}}(\rho)$ is quite essential to our approach here, since it is a central part in the construction of the boundary maps, the natural transformation of the two functors and also the natural homotopy.

Once we've identified the source of our problem, it is easier to think of solutions. What I imagine is a possible improvement to the current framework is to consider a pair of a sheaf and a cosheaf on a simplicial complex K. Intuitively, the data of both a sheaf and a cosheaf provides maps from both directions, thus is at least possible to solve our problem. Of course, we cannot choose an arbitrary sheaf $\mathcal{S}$ and cosheaf $\mathcal{C}$, because then there would be no relation between $\mathcal{S}(\sigma)$ and $\mathcal{C}(\sigma)$ and we cannot use them to construct the weight map $w(\rho)$. What we need is some compatibility relation between the sheaf $\mathcal{S}$ and the cosheaf $\mathcal{C}$. For any path

$$\rho = \sigma_1 \triangleleft \tau_1 \triangleright \cdots \triangleright \sigma_m \triangleleft \tau_m$$

suppose we want to construct the weight map

$$w_{\mathcal{C}}(\rho) : \mathcal{C}(\sigma_1) \to \mathcal{C}(\tau_m)$$

To use the sheaf data $\mathcal{S}$ we need maps

$$\mu_i : \mathcal{C}(\sigma_i) \to \mathcal{S}(\sigma_i),\ \nu_i : \mathcal{S}(\tau_i) \to \mathcal{C}(\tau_i)$$

where we can define

$$w_{\mathcal{C}}(\rho) := \nu_m \circ \mathcal{S}(\sigma_m \leq \tau_m) \circ \mu_m \circ \mathcal{C}(\sigma_m \leq \tau_{m-1}) \circ \cdots \circ \mathcal{C}(\sigma_2 \circ \tau_1) \circ \nu_1 \circ \mathcal{S}(\sigma_1 \leq \tau_1) \circ \mu_1$$

We might further requires some compatibility relation of the maps μ_i, ν_i as well. Any such compatibility requirement of μ_i and ν_i must say something about the following diagramme,

$$\begin{array}{ccc} \mathcal{C}(\sigma_i) & \xrightarrow{\mu_i} & \mathcal{S}(\sigma_i) \\ {\scriptstyle \mathcal{C}(\sigma_i \leq \tau_i)} \big\uparrow & & \big\downarrow {\scriptstyle \mathcal{S}(\sigma_i \leq \tau_i)} \\ \mathcal{C}(\tau_i) & \xleftarrow[\nu_i]{} & \mathcal{S}(\tau_i) \end{array}$$

I have not worked out the details of such an approach, and I do not know whether other constructions, such as $\psi_\bullet, \psi_\bullet$, would work nicely in such a framework. However, I imagine such an approach at least possible.

References

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