

# Algebraic Investigations of Intuitionistic Provability Logic and Paradoxes

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December 29, 2020

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## 1 Intuitionistic Magari Algebra

We first describe what is a Magari algebra in intuitionistic set up. In the following text we use  $\mathbb{H}, \mathbb{B}$  to denote a Heyting algebra and a Boolean algebra respectively. A Heyting algebra  $\mathbb{H}$  is a lattice where we have an exponential adjoint to meet,

$$a \wedge b \leq c \Leftrightarrow a \leq b \rightarrow c, \quad \forall a, b, c \in \mathbb{H}$$

It is well-known that any Heyting algebra  $\mathbb{H}$  is distributive. We will also assume that any Heyting or Boolean algebra in the following texts are bounded.

In algebraic context, a modal operator is an endofunctor  $\Box : \mathbb{H} \rightarrow \mathbb{H}$  which preserves finite limits,

$$\text{(NEC):} \quad \Box 1 = 1$$

$$\text{(K):} \quad \Box(a \wedge b) = \Box a \wedge \Box b$$

Such a pair  $\langle \mathbb{H}, \Box \rangle$  will be called (*intuitionistic*) *modal algebra*. Notice in a Heyting algebra, the less than relation can be defined equationally using meet, hence K implies monotonicity of  $\Box$ .

**Definition 1.** A modal algebra  $\langle \mathbb{H}, \Box \rangle$  is called (intuitionistic) Magari algebra if for all  $a \in \mathbb{H}$  we have

$$\text{(GL):} \quad \Box(\Box a \rightarrow a) = \Box a$$

In the following texts, modal algebra or Magari algebra will always be intuitionistic, unless otherwise specified explicitly. The aim of introducing Magari algebra is to prove various “paradoxical” results in arithmetic or syntax theory in a simple algebraic setting, in order to show that actually any paradoxes “reduced to Löb’s theorem”. But before doing that, let’s first see some examples of Magari algebra. First notice that for any Heyting algebra  $\mathbb{H}$  there is a trivial Magari algebra structure.

**Definition 2.** For any Heyting algebra  $\mathbb{H}$ , the trivial Magari algebra structure  $\langle \mathbb{H}, \Box \rangle$  is given by defining

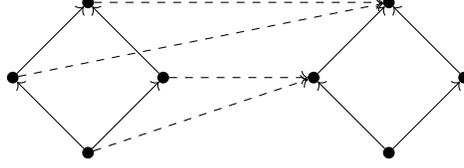
$$\Box a = 1$$

for any  $a \in \mathbb{H}$ .

First notice that this definition indeed makes  $\Box$  into a meet-preserving operator, because we have  $\Box 1 = 1$  and for any  $a, b \in \mathbb{H}$  we have  $\Box(a \wedge b) = \Box a \wedge \Box b = 1$ . In addition, for any  $a \in \mathbb{H}$ ,  $\Box a = 1$  implies  $\Box a \rightarrow a = 1 \rightarrow a = a$ . Then the scheme

$$\Box(\Box a \rightarrow a) = \Box a$$

is valid for any  $a \in \mathbb{H}$ . Thus, the above defined trivial Magari algebra is indeed well-defined. The following is an example of a non-trivial Magari algebra structure when  $\mathbb{H} = \mathbb{B}_4$ , the four element Boolean algebra. The map of the  $\Box$  operator is depicted below:



The mapping of the  $\Box$  operator is given by the dashed lines. It is not hard to verify that this indeed gives us a Magari algebra structure, and it is the only non-trivial structure (up to isomorphism) in  $\mathbb{B}_4$ .

We show some basic properties of Magari algebra. First, we show that the modal axiom S4 is valid in any Magari algebra.

$$(S4): \quad \Box a \leq \Box \Box a$$

**Lemma 1 (S4).** *S4 is valid in every Magari algebra  $\langle \mathbb{H}, \Box \rangle$ .*

*Proof.* Notice that we have the following computation,

$$\Box(\Box a \wedge a) \rightarrow \Box a \wedge a = \Box \Box a \wedge \Box a \rightarrow \Box a \wedge a = \Box \Box a \wedge \Box a \rightarrow a$$

The first equality holds by K and the second holds by (3) of Lemma 8 in the Appendix. It follows that

$$a \leq \Box(\Box a \wedge a) \rightarrow \Box a \wedge a$$

By monotonicity, GL, and K,

$$\Box a \leq \Box(\Box(\Box a \wedge a) \rightarrow \Box a \wedge a) = \Box(\Box a \wedge a) = \Box \Box a \wedge \Box a$$

This actually shows that  $\Box \Box a \wedge \Box a = \Box a$  and thus  $\Box a \leq \Box \Box a$ . □

A closely related lemma is presented as follows.

**Lemma 2.** *In a Magari algebra  $\langle \mathbb{H}, \Box \rangle$ , for any  $a \in \mathbb{H}$  we have*

$$\Box a = \Box \Box a \Leftrightarrow \Box a = 1$$

*Proof.* The “if” part is trivial. Suppose  $\Box a = \Box\Box a$ . By GL we have

$$\Box(\Box\Box a \rightarrow \Box a) = \Box\Box a$$

$\Box\Box a = \Box a$  implies  $\Box\Box a \rightarrow \Box a = 1$ . Hence we have

$$\Box\Box a = \Box a = \Box 1 = 1$$

This gives the proof.  $\square$

**Lemma 3.** *In any Magari algebra  $\langle \mathbb{H}, \Box \rangle$  and any  $a \in \mathbb{H}$ , we use  $\Box^k a$  to denote the result of applying the  $\Box$  operator  $k$  times iteratively to  $a$ . Then by Lemma 1 we have the following long sequence,*

$$\Box a \leq \Box\Box a \leq \dots \leq \Box^k a \leq \dots$$

*If this sequence every stablizes at some point, i.e. there exists  $k \in \mathbb{N}_+$  such that*

$$\Box^k a = \Box^{k+1} a$$

*then it must be the case that  $\Box^k a = 1$ , and hence  $\Box^{k+n} a = 1$  for any  $n \in \mathbb{N}$ . In particular, before the sequence stablizes, all the inequalities are strict.*

*Proof.* It’s a direct application of Lemma 1 and Lemma 2.  $\square$

Next we show the formalized version of Löb’s theorem and the informal version are equivalent. First we define the notion of *algebra* of  $\Box$  in any modal algebra  $\langle \mathbb{H}, \Box \rangle$ .

**Definition 3.** *An algebra of  $\Box$  in any modal algebra  $\langle \mathbb{H}, \Box \rangle$  is an object  $a \in \mathbb{H}$  such that*

$$\Box a \leq a$$

*The set of all algebras of  $\Box$  in  $\mathbb{H}$  is denoted as  $\mathbf{Alg}_\Box(\mathbb{H})$ .*

A simple observation shows that  $\mathbf{Alg}_\Box(\mathbb{H})$  is non-empty, because by definition we have  $\Box 1 = 1$  and thus  $\Box 1 \leq 1$ , which shows  $1 \in \mathbf{Alg}_\Box(\mathbb{H})$ . We say that  $\mathbf{Alg}_\Box(\mathbb{H})$  is *trivial* if it only contains 1. Let me state some more simple properties of the set  $\mathbf{Alg}_\Box(\mathbb{H})$ .

**Lemma 4.** *Let  $\langle h, \Box \rangle$  be a modal algebra and let  $\mathbf{Alg}_\Box(h)$  be the set of algebras of  $\Box$  in  $\mathbb{H}$ , then we have:*

- (1)  $\mathbf{Alg}_\Box(\mathbb{H})$  is closed under meet;
- (2) if  $x \in \mathbf{Alg}_\Box(\mathbb{H})$  then so does  $x \wedge \Box x, x \wedge \Box x \wedge \Box\Box x, \dots$ ;

*Proof.* (1) Suppose we have  $x, y \in \mathbf{Alg}_\Box(\mathbb{H})$ . Then by definition and K,

$$\Box(x \wedge y) = \Box x \wedge \Box y \leq x \wedge y$$

Hence we have  $x \wedge y \in \mathbf{Alg}_\Box(\mathbb{H})$ .

(2) Suppose  $x \in \mathbf{Alg}_\square(\mathbb{H})$ , viz.  $\square x \leq x$ . Then we have

$$\square(x \wedge \square x) = \square\square x$$

By monotonicity,  $\square x \leq x$  implies that  $\square\square x \leq \square x$ . Hence we have

$$\square(x \wedge \square x) = \square\square x \leq \square x = x \wedge \square x$$

This shows that indeed  $x \wedge \square x \in \mathbf{Alg}_\square(\mathbb{H})$ . Now notice that by K

$$x \wedge \square x \wedge \square(x \wedge \square x) = x \wedge \square x \wedge \square x \wedge \square\square x = x \wedge \square x \wedge \square\square x$$

Hence the same argument gives us  $x \wedge \square x \wedge \square\square x \in \mathbf{Alg}_\square(\mathbb{H})$ . Other cases follows inductively.

(3)

The following lemma shows that to some extent we can use the properties of  $\mathbf{Alg}_\square(\mathbb{H})$  to completely characterize Magari algebra (with the help of S4).

**Lemma 5.** *Suppose  $\langle \mathbb{H}, \square \rangle$  is a modal algebra. Then the following two conditions are equivalent:*

1. *GL is valid in  $\langle \mathbb{H}, \square \rangle$ ;*
2. *S4 is valid in  $\langle \mathbb{H}, \square \rangle$  and  $\mathbf{Alg}_\square(\mathbb{B})$  is trivial, viz. only contains 1.*

*Proof.* (1)  $1 \Rightarrow 2$ : In Lemma 1 we've already shown that S4 is implied by GL, hence we only need to show that  $\mathbf{Alg}_\square(\mathbb{H})$  is trivial. Suppose  $\square a \leq a$ , then we have

$$\square a \rightarrow a = 1$$

By GL it follows

$$\square a = \square(\square a \rightarrow a) = \square 1 = 1$$

However, we also know that

$$\square a \leq a$$

It follows that  $a = 1$ , hence  $\mathbf{Alg}_\square(\mathbb{H})$  is indeed trivial.

(2)  $2 \Rightarrow 1$ : To simplify notation, we define the following two modalities,

$$\nabla x := \square x \rightarrow x, \quad \Delta x := \square \nabla x \rightarrow \square x = \square(\square x \rightarrow x) \rightarrow \square x$$

We then proceed to show that in any modal algebra where S4 is valid,  $\Delta x \in \mathbf{Alg}_\square(\mathbb{H})$  for any  $x \in \mathbb{H}$ . Once this is shown, if in addition we know  $\mathbf{Alg}_\square(\mathbb{H})$  is trivial, then  $\Delta x = 1$  for any  $x$ . Notice that for any  $x$  we have

$$x \leq \square x \rightarrow x$$

Hence, by monotonicity we have

$$\square x \leq \square(\square x \rightarrow x)$$

If now  $\Delta x = 1$ , we must have

$$\Box(\Box x \rightarrow x) \leq \Box x$$

Together they imply that GL is valid. Hence, we only need to prove  $\Delta x \in \mathbf{Alg}_{\Box}(\mathbb{H})$ , viz.

$$\Box \Delta x \leq \Delta x$$

for any  $x \in \mathbb{H}$ . First, we have the following computation,

$$\begin{aligned} \Box \Delta x \wedge \Box \Box \nabla x &= \Box((\Box \nabla x \rightarrow \Box x) \wedge \Box \nabla x) \\ &= \Box(\Box \nabla x \wedge \Box x) \\ &= \Box \Box x \wedge \Box \Box \nabla x \end{aligned}$$

The second equality holds because (2) of Lemma 8 in Appendix. Similarly, we also have

$$\begin{aligned} \Box \nabla x \wedge \Box \Box x &= \Box((\Box x \rightarrow x) \wedge \Box x) \\ &= \Box(\Box x \wedge x) \\ &= \Box x \wedge \Box \Box x \end{aligned}$$

By S4, we actually know  $\Box x \wedge \Box \Box x = \Box x$ . It also follows from S4 and monotonicity that

$$\begin{aligned} \Box \Delta x \wedge \Box \nabla x &\leq \Box \Delta x \wedge \Box \Box \nabla x \\ &= \Box(\Box x \wedge \Box \nabla x) \\ &\leq \Box(\Box \Box x \wedge \Box \nabla x) \\ &= \Box(\Box x \wedge \Box \Box x) \\ &= \Box \Box x \end{aligned}$$

Hence we also have

$$\Box \Delta x \wedge \Box \nabla x \leq \Box \Box x \wedge \Box \nabla x = \Box x \wedge \Box \Box x = \Box x$$

By the adjunction, we have

$$\Box \Delta x \leq \Box \nabla x \rightarrow \Box x = \Delta x$$

This actually shows that if S4 is valid in  $\langle \mathbb{H}, \Box \rangle$ ,  $\Delta x \in \mathbf{Alg}_{\Box}(\mathbb{H})$  for any  $x \in \mathbb{H}$ . Then straight forwardly if  $\mathbf{Alg}_{\Box}(\mathbb{H})$  is trivial then GL is valid.  $\square$

Here we have shown that in a Magari algebra the only element that can have the property  $\Box x \leq x$  is the terminal object 1, and this property, together with S4, actually completely characterizes Magari algebra.

## 2 Analysis of Propositional Paradoxes

In this section, we use the intuitionistic Magari algebra structure described in the first section to show some results of propositional paradoxes. Here, by propositional paradoxes I mean those whose formulation do not involve (internal) quantifiers. We first show a simple lemma, and we prove that, surprisingly, almost all propositional paradoxes are direct implication of this fact.

**Lemma 6.** *In any non-degenerate (viz.  $0 \neq 1$ ) Magari algebra  $\langle \mathbb{H}, \Box \rangle$ , we must have*

$$\Box 0 \neq 0$$

*Proof.* Suppose  $\Box 0 = 0$ , then we have  $\Box 0 \rightarrow 0 = 1$ . It follows that

$$\Box(\Box 0 \rightarrow 0) = \Box 1 = 1$$

This shows that  $\Box(\Box 0 \rightarrow 0) \neq \Box 0$  unless  $0 = 1$ , and thus GL fails. Hence in any non-degenerate Magari algebra  $\langle \mathbb{H}, \Box \rangle$ , we must have  $0 < \Box 0$ , strictly.  $\square$

In the following texts I will always assume that the Heyting algebra  $\mathbb{H}$  is non-degenerate, unless otherwise specified. I'll show that the fact stated in Lemma 6 alone could explain almost all propositional paradoxes. For example, we can prove the liar result as follows:

**Theorem 1 (Liar).** *If  $\langle \mathbb{H}, \Box \rangle$  is a Magari algebra then the axiom scheme*

$$\Box a = a$$

*cannot be valid.*

*Proof.* We've already shown that we can never have  $\Box 0 \neq 0$ , hence  $\Box a = a$  cannot be true for all elements in a Magari algebra  $\langle \mathbb{H}, \Box \rangle$ .  $\square$

Actually, this proof gives a slightly stronger result.

**Theorem 2 (Montague).** *If  $\langle \mathbb{H}, \Box \rangle$  is a Magari algebra then the axiom scheme*

$$\Box a \leq a$$

*cannot be valid.*

*Proof.* The same proof in the case of Liar suffices.  $\square$

**Theorem 3 (Gödel's First Incompleteness Theorem).** *In a Magari algebra  $\langle \mathbb{H}, \Box \rangle$ , if for any  $x \in \mathbb{H}$  we have*

$$x = 1 \text{ iff } \Box x = 1$$

*then  $\mathbb{H}$  cannot be the two element Heyting algebra  $\mathbb{H}_2$  (it's actually Boolean, viz.  $\mathbb{H}_2 = \mathbb{B}_2$ ).*

*Proof.* Again, this is a very simple application of Lemma 6. In  $\mathbb{H}_2$ , since we cannot have  $\Box 0 = 0$ , the only choice is that  $\Box 0 = 1$ , which is inconsistent with the claim that  $x = 1$  iff  $\Box x = 1$ .  $\square$

This is formulated as Gödel's First Incompleteness Theorem because Theorem 3 shows in a Magari algebra  $\langle \mathbb{H}, \Box \rangle$  we must have elements which are neither 1 nor 0, viz. neither provable nor refutable.

Actually, we can prove the following stronger version of Theorem 3.

**Theorem 4 (Infinite Incompleteness Theorem).** *In a Magari algebra  $\langle \mathbb{H}, \Box \rangle$ , if for any  $x \in \mathbb{H}$  we have*

$$x = 1 \text{ iff } \Box x = 1$$

*then  $\mathbb{H}$  cannot be a finite Heyting algebra.*

*Proof.* In Lemma 3 we have shown that there is the following infinite sequence in  $\mathbb{H}$  for any Magari algebra  $\langle \mathbb{H}, \Box \rangle$ ,

$$0 \leq \Box 0 \leq \dots \leq \Box^k 0 \leq \dots$$

If  $\mathbb{H}$  is finite, then we cannot have all these inequalities to be strict; let  $k$  be the smallest number such that  $\Box^k 0 = \Box^{k+1} 0$ , and by Lemma 3 it follows that

$$\Box^{k+n} 0 = 1, \forall n \in \mathbb{N}$$

In particular,  $\Box^k 0 = 1$ . Since  $k$  is the smallest such number, we must have that

$$\Box^{k-1} 0 \neq \Box^k 0$$

and hence

$$\Box^{k-1} 0 \neq 1, \quad \Box \Box^{k-1} 0 = \Box^k 0 = 1$$

The above argument actually shows that in any finite Magari algebra, we must have an element  $a \neq 1$  such that  $\Box a = 1$ . Hence, the condition  $x = 1$  iff  $\Box x = 1$  cannot be satisfied in any finite Mageri algebra.  $\square$

**Theorem 5 (Postcard).** *If  $\langle \mathbb{H}, \Box \rangle$  is a Magari algebra, then the scheme*

$$\Box \Box a = a$$

*is inconsistent.*

*Proof.* Again, this simply follows from Lemma 6. If we have  $0 = \Box \Box 0$  and we also know that  $\Box 0 \leq \Box \Box 0$  in a Magari algebra, then it follows  $\Box 0 = 0$ . Lemma 6 shows this cannot be true in any Magari algebra.  $\square$

**Theorem 6 (Thomason).** *If  $\langle \mathbb{H}, \Box \rangle$  is a Magari algebra and it validates the scheme*

$$\Box(\Box a \rightarrow a) = 1$$

*then it must be a trivial Magari algebra.*

*Proof.* Trivial.  $\square$

### 3 Terminal Completeness

A classical result given by Solovay states that the (classical) modal logic GL is arithmetic complete. We may prove a similar result using algebraic terms. First, let us define homomorphisms between modal algebras.

**Definition 4.** A homomorphism  $h : \langle \mathbb{H}_1, \square_1 \rangle \rightarrow \langle \mathbb{H}_2, \square_2 \rangle$  between two modal algebras is a lattice homomorphism  $h : \mathbb{H}_1 \rightarrow \mathbb{H}_2$  which in addition commutes with the modal operator,

$$h(\square_1 x) = \square_2(hx), \quad \forall x \in \mathbb{H}_1$$

In the following texts we will in general not distinguish notationally the two different modal operators in the two modal algebra, if there will be no confusion. The reader should always keep in mind which modal operator is in which modal algebra.

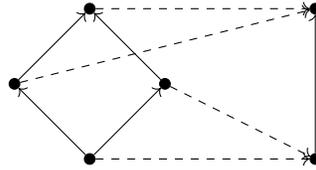
**Definition 5 (Terminally Complete Magari Algebra).** A Magari algebra  $\langle \mathbf{H}, \square \rangle$  is said to be terminally complete, if for any Magari algebra  $\langle \mathbb{H}, \square \rangle$  we have the following property

$$\forall x \in \mathbb{H}. (\forall h \in \text{Hom}(\langle \mathbb{H}, \square \rangle, \langle \mathbf{H}, \square \rangle)). h(x) = 1 \Rightarrow x = 1$$

In other words, if  $\langle \mathbf{H}, \square \rangle$  is terminally complete, then for any Magari algebra  $\langle \mathbb{H}, \square \rangle$ , there is no element  $x \in \mathbb{H}$  and homomorphism  $h : \langle \mathbb{H}, \square \rangle \rightarrow \langle \mathbf{H}, \square \rangle$  such that

$$x \neq 1 \ \& \ h(x) = 1$$

To show this definition is non-trivial, we observe that not all Magari algebra, e.g.  $\mathbb{H}_2$ , is terminally complete. Suppose we have a Magari algebra in which  $\square 0 \neq 1$ , e.g. the Magari algebra  $\langle \mathbb{B}_4, \square \rangle$  I show previously. There is a homomorphism between the two depicted by the following graph:



where the homomorphism  $h : \langle \mathbb{B}_4, \square \rangle \rightarrow \langle \mathbb{H}_2, \square \rangle$  is given by the dashed lines. It is straight forward to verify that  $h$  is indeed a well-defined homomorphism, and we have

$$h(\square 0) = \square h(0) = \square 0 = 1$$

However,  $\square 0$  fails to be 1 in  $\langle \mathbb{B}_4, \square \rangle$ , and this shows that  $\langle \mathbb{H}_2, \square \rangle$  cannot be terminally complete. To state a more general result, we first define the notion of *degree* of a Magari algebra:

**Definition 6.** In any Magari algebra  $\langle \mathbb{H}, \Box \rangle$ , the degree of  $\langle \mathbb{H}, \Box \rangle$ , denoted as  $\deg(\langle \mathbb{H}, \Box \rangle)$  (or simply  $\deg(\mathbb{H})$  when there's no confusion about which Magari algebra structure on  $\mathbb{H}$  we are talking about), is defined to be the smallest natural number  $n$  such that

$$\Box^n 0 = 1$$

If no such  $n$  exists, then we simply say that

$$\deg(\mathbb{H}) = \infty$$

**Lemma 7.** Suppose  $h : \langle \mathbb{H}_1, \Box_1 \rangle \rightarrow \langle \mathbb{H}_2, \Box_2 \rangle$  is a Magari algebra homomorphism, then we must have

$$\deg(\mathbb{H}_1) \geq \deg(\mathbb{H}_2)$$

*Proof.* Notice that for any  $n \in \mathbb{N}$  we have

$$h(\Box^n 0) = \Box^n h(0) = \Box^n 0$$

Suppose  $\deg(\mathbb{H}_1) = n_1$ , that is

$$\Box^{n_1} 0 = 1$$

Now if  $\deg(\mathbb{H}_2) > \deg(\mathbb{H}_1)$ , we would have

$$h(1) = h(\Box^{n_1} 0) = \Box^{n_1} 0 \neq 1$$

Hence,  $h$  is not a Magari algebra homomorphism. This shows that Magari algebra homomorphism cannot go from lower degrees to higher degrees.

**Theorem 7.** Any Magari algebra  $\langle \mathbb{H}, \Box \rangle$  of finite degree is not terminally complete.

*Proof.* The proof use the fact that the Lindenbaum–Tarski construction of the (intuitionistic) modal logic GL results in the free Magari algebra. I do not give a proof here but it should be a standard implication from the study of algebraic logic. Call the free Magari algebra with no generators  $\langle \mathbb{F}_0, \Box \rangle$  (it's not empty because we always need element  $0, 1$  in a Heyting algebra). Since  $\langle \mathbb{F}_0, \Box \rangle$  is the free algebra with no generator, for any Magari algebra  $\langle \mathbb{H}, \Box \rangle$  we have a unique homomorphism

$$h : \langle \mathbb{F}_0, \Box \rangle \rightarrow \langle \mathbb{H}, \Box \rangle$$

In Appendix B we have explicitly constructed Magari algebra of degree  $n$  for any finite  $n \in \mathbb{N}$ . By Lemma 7 it follows that we must have

$$\deg(\mathbb{F}_0) \geq n, \forall n \in \mathbb{N}$$

and hence it must be the case that

$$\deg(\mathbb{F}_0) = \infty$$

Now consider  $h : \langle \mathbb{F}_0, \Box \rangle \rightarrow \langle \mathbb{H}, \Box_{\mathbb{H}} \rangle$  where  $\deg(\mathbb{H}) = n$  is finite, we actually have an element  $\Box^n 0 \neq 1 \in \mathbb{F}_0$ , but

$$h(\Box^n 0) = \Box_{\mathbb{H}}^n 0 = 1$$

This shows Magari algebra with finite degree cannot be terminally complete.  $\square$

## Appendix A

In this appendix we prove some simple computation in Heyting algebra where we have used in the previous proofs.

**Lemma 8.** *The following are all valid computations in any Heyting algebra:*

- (1)  $1 \rightarrow a = a$ ;
- (2)  $a \wedge (a \rightarrow b) = a \wedge b$ ;
- (3)  $a \wedge b \rightarrow b \wedge c = a \wedge b \rightarrow c$ ;
- (4)  $a \leq c$  implies that  $c \rightarrow b \leq a \rightarrow b, b \rightarrow a \leq b \rightarrow c$ ;

*Proof.* (1) By the adjunction we have

$$z \leq 1 \rightarrow a \Leftrightarrow z \wedge 1 \leq a \Leftrightarrow z \leq a$$

Thus by Yoneda, we have  $1 \rightarrow a = a$ .

- (2) First it is easy to see that  $a \wedge (a \rightarrow b) \leq a$ . We also have

$$a \rightarrow b \leq a \rightarrow b$$

Hence by the adjunction we have

$$a \wedge (a \rightarrow b) \leq b$$

Then it follows that

$$a \wedge (a \rightarrow b) \leq a \wedge b$$

On the other hand, we also know

$$a \wedge b \leq a$$

In addition, we also have

$$a \wedge (a \wedge b) = a \wedge b \leq b$$

Hence by the adjunction again,

$$a \wedge b \leq a \rightarrow b$$

This shows we also have

$$a \wedge b \leq a \wedge (a \rightarrow b)$$

In a poset, the above means indeed we have

$$a \wedge (a \rightarrow b) = a \wedge b$$

(3) On the one hand, by 2 we have

$$(a \wedge b) \wedge (a \wedge b \rightarrow b \wedge c) = (a \wedge b) \wedge (b \wedge c) = a \wedge b \wedge c \leq c$$

By the adjunction we have

$$a \wedge b \rightarrow b \wedge c \leq a \wedge b \rightarrow c$$

On the other hand, by 2 again we have

$$(a \wedge b) \wedge (a \wedge b \rightarrow c) = a \wedge b \wedge c \leq b \wedge c$$

Hence by the adjunction it follows

$$a \wedge b \rightarrow c \leq a \wedge b \rightarrow b \wedge c$$

The above shows that we have

$$a \wedge b \rightarrow b \wedge c = a \wedge b \rightarrow c$$

(4) The fact that  $a \leq c$  implies  $b \rightarrow a \leq b \rightarrow c$  simply follows from that fact that in any Cartesian closed category  $\mathcal{C}$ , for any object  $B$  in  $\mathcal{C}$ ,  $(-)^B : \mathcal{C} \rightarrow \mathcal{C}$  is a covariant functor. We may prove the other fact as follows. We know that

$$a \wedge (c \rightarrow b) \leq c \wedge (c \rightarrow b) = c \wedge b \leq b$$

Hence by the adjunction we must have

$$c \rightarrow b \leq a \rightarrow b$$

## Appendix B

In this section we explicitly construct Magari algebra of any finite degree  $n \in \mathbb{N}$ . We use  $\mathbb{B}_{2^n}$ , the Boolean algebra of  $2^n$  elements, as our base Heyting algebra and we'll defined a Magari algebra structure  $\langle \mathbb{B}_{2^n}, \square_n \rangle$  of degree  $n$  for each  $n \in \mathbb{N}$ . We construct this structure inductively:

(1) When  $n = 0$ , we have a degenerate Boolean algebra  $\mathbb{B}_{2^0} = \mathbb{B}_1$ , where obviously  $\square_0 1 = 1$  is the only choice. Notice that in  $\mathbb{B}_{2^0}$ , we have  $0 = 1$ . Hence indeed we have

$$\text{deg}(\langle \mathbb{B}_{2^0}, \square_0 \rangle) = 0$$

(2) Suppose we have already defined a Magari algebra structure  $\langle \mathbb{B}_{2^n}, \square_n \rangle$  such that

$$\text{deg}(\mathbb{B}_{2^n}) = n$$

We use this data to give a Magari algebra structure on  $\mathbb{B}_{2^{n+1}}$ . First, notice that in  $\mathbb{B}_{2^{n+1}}$  we can write its elements of the following form,

$$\mathbb{B}_{2^{n+1}} \cong \mathbb{B}_{2^n} \times \mathbb{B}_2 \cong \{ (x, i) \mid x \in \mathbb{B}_{2^n}, i = 0, 1 \}$$

where all lattice-theoretic operation is defined to be coordinate-wise. For a proof of this fact see Lemma ?? in the Appendix. Then we may define a Magari algebra structure  $\langle \mathbb{B}_{2^{n+1}}, \square_{n+1} \rangle$  by specifying,

$$\square_{n+1}(x, 0) = (0, 1), \quad \square_{n+1}(x, 1) = (\square_n x, 1), \quad \forall x \in \mathbb{B}_{2^n}$$

It's easy to see that  $\square_{n+1}$  is a well-defined modal operator on  $\mathbb{B}_{2^{n+1}}$ : for any  $(x, i), (y, j)$ , if one of  $i, j$  is 0, then we have

$$(x, i) \wedge (y, j) = (x \wedge y, 0)$$

and by definition

$$\square_{n+1}((x, i) \wedge (y, j)) = \square_{n+1}(x \wedge y, 0) = (0, 1)$$

On the other hand,

$$(0, 1) \leq (\square_n x, 1), \quad (0, 1) \leq (\square_n y, 1)$$

Hence no matter  $i$  or  $j$  equals to 0 we have

$$\square_{n+1}(x, i) \wedge \square_{n+1}(y, j) = (0, 1) = \square_{n+1}((x, i) \wedge (y, j))$$

If both  $i, j = 1$ , then it simply follows from the induction hypothesis that  $\square_n$  is a well-defined modal operator,

$$\begin{aligned} \square_{n+1}((x, 1) \wedge (y, 1)) &= \square_{n+1}(x \wedge y, 1) \\ &= (\square_n(x \wedge y), 1) \\ &= (\square_n x \wedge \square_n y, 1) \\ &= (\square_n x, 1) \wedge (\square_n y, 1) \\ &= \square_{n+1}(x, 1) \wedge \square_{n+1}(y, 1) \end{aligned}$$

Hence,  $\square_{n+1}$  is indeed a well-defined modal operator. Now for GL, we have the following computation,

$$\square_{n+1}(x, 0) \rightarrow (x, 0) = (0, 1) \rightarrow (x, 0) = (1, 0) \vee (x, 0) = (1, 0)$$

$$\square_{n+1}(\square_{n+1}(x, 0) \rightarrow (x, 0)) = \square_{n+1}(1, 0) = (0, 1) = \square_{n+1}(x, 0)$$

This shows that GL is valid for any  $(x, 0)$ . Now for  $(x, 1)$ , by induction hypothesis  $\langle \mathbb{B}_{2^n}, \square_n \rangle$  is a Magari algebra and we have

$$\square_{n+1}(x, 1) \rightarrow (x, 1) = (\square_n x, 1) \rightarrow (x, 1) = (\square_n x \rightarrow x, 1)$$

$$\begin{aligned} \square_{n+1}(\square_{n+1}(x, 1) \rightarrow (x, 1)) &= \square_{n+1}(\square_n x \rightarrow x, 1) \\ &= (\square_n(\square_n x \rightarrow x), 1) \\ &= (\square_n x, 1) = \square_{n+1}(x, 1) \end{aligned}$$

This shows that indeed  $\langle \mathbb{B}_{2^{n+1}}, \square_{n+1} \rangle$  is a well-defined Magari algebra.

Now notice that according to the definition of the structure  $\langle \mathbb{B}_{2^n}, \square_n \rangle$ , if we know that

$$\deg(\mathbb{B}_{2^n}) = k$$

then we would have

$$\square_{n+1}(0, 0) = (0, 1), \quad \square_{n+1}^s(0, 0) = (\square_n^{s-1}0, 1)$$

Hence it follows that

$$\deg(\mathbb{B}_{2^{n+1}}) = k + 1$$

We've already shown that

$$\deg(\mathbb{B}_1) = 0$$

It then follows

$$\deg(\mathbb{B}_{2^n}) = n, \quad \forall n \in \mathbb{N}$$

We have thus explicitly constructed Magari algebra of any finite degree. We can now prove the above theorem.