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A Structural Study of Information in a Logical Perspective

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中文摘要

本论文旨在使用范畴论的语言，从逻辑的角度对“信息景观”的内部结构进行研究。着眼于哲学和逻辑上的应用，在本文中信息结构将被理解为那些支持多主体动态模态系统的语义模型。常见的信息结构包括克里普克框架、拓扑空间、邻域框架、代数模型等。这些不同类型的数学结构可以看作是信息的不同表示方式，也即我们所探讨的信息景观当中的不同元素。

本论文旨在首先采用范畴论的框架，精确地描述所有支持此类逻辑推理的信息模型，揭示它们背后共同的数学结构。从哲学的层面说，它们共同的数学结构可以看作是对信息结构更加本质地描述，这有助于我们理解信息在日常推理中扮演的角色。从技术的层面上讲，这会将模态逻辑看似不同的扩张，包括静态与动态扩张，与其各种各样的语义解释统一起来。其次，我们将借助范畴之间的函子来研究所有不同类型的信息结构之间的相互关联。不同的函子将把信息景观中的各个元素连接起来，构成一个整体，帮助我们更加深入地理解信息景观的整体结构，以及不同信息层次对象之间如何相互作用。从观念层面来看，该项目的最终目标是对信息景观的内部结构进行系统地阐释，以期更好地理解信息在我们有关知识、信念等推理中所扮演的作用。从技术层面来看，这项工作也统一了一类非常广泛的动态模态逻辑片段的语义模型。我们期望这项将观念阐述与技术发展相结合的工作能够对现有逻辑、哲学中相关问题的研究有所帮助。

关键词：信息结构；模态逻辑；模态强度；群体主体；动态逻辑；范畴论

ABSTRACT

The thesis aims to give a structural study of the "landscape of information" from a logical perspective, using the language of category theory. With an eye towards philosophical and logical applications, the general type of information structures we consider are those supporting the interpretation of certain (extensions of) modal systems. Common structures of this type include Kripke frames, topological spaces, neighbourhood frames, or various other algebraic models. These different types of mathematical structures are considered as different information levels, which can be identified as different elements of the landscape of information.

The project set out by this thesis then aims to first provide a precise description of the common mathematical background for all such types of models which characterise information structures that support logic reasonings, using a categorical point of view. In particular, we describe the interpretation of various fragments of modal logic, including modal dependence, group structure, and logical dynamics, for all types of semantics in a uniform way. We will secondly look at the interactions between all these different types of semantics by considering functorial transformations between them. These transformations consist of various connections within the information landscape, and help us obtain a more conceptual understanding of how different levels of information structure interplay with each other. We hope such technical development would be useful for both philosophers and logicians working in related fields.

Keywords: Information Structure; Modal Logic; Modal Dependence; Group Structure; Logical Dynamics; Category Theory

CONTENTS

ABBREVIATIONS

CABA	Complete Atomic Boolean Algebra
CABAO	Complete Atomic Boolean Algebra with Operators
DEL	Dynamic Epistemic Logic
EL	Evidence Logic
LCD	Logic of Continuous Dependence
LFD	Logic of Functional Dependence
PAL	Public Announcement Logic
CON	Contraction Update Logic
PRO	Product Type Update Logic
COP	Coproduct Type Update Logic
Set	The Category of Sets
Kr	The Category of Kripke Frames
Pre	The Category of Preorders
Eqv	The Category of Equivalence Relations
Top	The Category of Topological Spaces
Nb	The Category of Neighbourhood Frames
Mon	The Category of Monotone Neighbourhood Frames
LMon	The Category of Left Exact Monotone Neighbourhood Frames

Chapter 1 Motivation and Introduction

In this chapter, we will explain in some detail the motivation behind the project, the problems we are going to address, and the methodology and mathematical framework we are going to adopt. Different sections within this chapter also serve as an extended outline of later chapters.

1.1 The Landscape of Information

There is a long tradition in philosophy and philosophical logic to use various different kinds of modal logics to study epistemological notions like knowledge and belief. The modern form of the classical epistemic logic, as we've known it today, is commonly acknowledged as initiated by von Wright's seminal work[1] in 1951. It first suggested to use a relational model to provide the semantics of epistemic modal logic. This was then extended by Hintikka's work[2] in 1962. There, the intuition of expressing various epistemic notions in terms of possible worlds was made much clearer. As such, it has served as the foundational text of epistemic logic ever since.

Over these past years, a great variety of modal systems based on other kinds of semantic models, both relational and non-relational ones, have been proposed to model more refined notion of knowledge and belief. Besides the pure epistemic logic with an uncertainty relation, we have more refined plausibility models that allow us to compare the likely-hood of the worlds so as to form belief; similar ideas to this already appeared very early on in the logic of conditionals[3] or related philosophical literature[4]. Much more recently we also have evidence logic embodying the neighbourhood semantics, which models evidence-based knowledge and belief[5]; or also topological models modelling epistemic knowledge and dependence[6-7]. Textbook-level introduction to most of the technical background and the philosophical ideas behind the above mentioned aspects of modal logic could can be found in[8-9].

Information comes into the picture — or even to say, integrated deeply into the picture — because there is a very natural information-theoretic reading of the above variety of epistemic and doxastic logical systems. All these different classes of models can be viewed as *different ways of representing or modelling information*, some coarser, others more refined; and the notion of knowledge and belief naturally arises in the corresponding level of the information structure. The activities of reasoning and forming knowledge or belief about the current situation one encounters, after all, are just different forms of *information handling* by rational agents. In practice, which way one are actually going to use to perform reasoning depending on the specific tasks (for a much more detailed discussion on this, see[10]).

From the perspective of information structure, the above mentioned different classes of models, or representations of information, naturally organise themselves into an ordered diagramme, in terms of how much informational content they reveal. From the top to bottom, we can put these different classes of models into a unified picture in the order of coarser to finer structures, as indicated by the following picture:

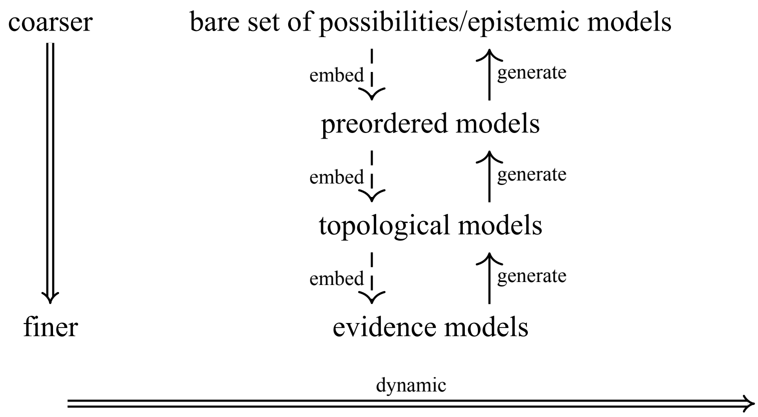

Figure 1.1 A Corner of the Landscape of Information

Such a diagramme could be thought of as a miniature of what we call *the landscape of information* (cf.[11]). From a logical perspective, the total landscape of information we would like to study is, roughly speaking, a collection of mathematical structures that carries informational content, so as for agents to form knowledge and belief, and reason about them.

Though being a corner of the landscape, many of the primary features we would like to investigate for the total landscape can already be seen from Figure 1.1. For instance, as we have seen, there are different levels of informational structure in terms of generality, and different levels are usually connected via certain transformations, e.g. the vertical arrows labelled as "embed" and "generate" in the above picture. However, the general landscape would not be a linear sequence of information structures. The shape of the diagramme would be much more involved. We also have the horizontal arrow labelled as "dynamic" at the very bottom. This suggests that, within each level of the information structure, all of them will support certain reasoning about the dynamics. We will come back to these points in the next section, where we address more specifically what problems we set out to study for the information landscape.

1.2 Problems at Hand

As explained in[11], generally speaking there will be two modes of research attitudes towards such a landscape: we can either view elements in it as *competitive* models against each other, and then the research goal in this case, naturally, is to determine which single one structure we should adopt for our general goal of representing informational content from a logical perspective; or if none of them is adequate, to find the *ultimate one* modelling of information that we should use. However, this approach is thought of as unrealistic, or at least, not useful, in our perspective: We have plenty of experiences in practice that, when dealing with various sorts of scenarios, different modellings would turn out to serve as the most adequate and most simple tool to solve the issue at hand (again, see relevant discussions in[10]). The natural counterpart of such an attitude then, is to take all of the information structures we meet in practice seriously, and try to determine whether we can find a common ground for all of them, and what inter-connections among them could be. This is the approach that will be adopted by us in this thesis, and the following part of this section is a more elaborate description of the concrete problems we are going to address in future texts.

1.2.1 Abstract Structure Behind the Landscape

As indicated in the title of the thesis, our general philosophy within this project is to adopt a structural perspective. The general goal is then to *conceptualise* the landscape of information, and find an appropriate mathematical framework to study it in a systematic fashion. Notice that, simply by putting these different classes of models into one picture ordered by coarseness is very far from a complete understanding of the common structure behind the landscape, and the connections within. To be more specific, at the very beginning of studying the landscape of information, we have already encountered a very deep problem concerning the structure of its elements:

Problem 1.1: Is there a common structure behind all these different classes of models that best reveals their abilities of both representing different levels of informational structures, and of supporting certain general kind of modal logic so that agents could reason about their knowledge and belief induced by such information content? ◀

In other worlds, Problem 1.1 asks for the common (mathematical) structures behind these classes of models that make them good candidates of representing semantic informational content. This problem is fundamental to all other problems we will address. To some extend, a positive answer to this problem is the very justification that we could *ever* put these different models into a same picture and compare them.

The common ground of these semantic models would also allow us to identify the correspondence between, on one hand, the mathematical structures of models, and on the other hand, certain linguistic features of the language of various kinds of modal logic based on those semantics. This correspondence, once identified, has the benefit of providing a unifying treatment of different extensions of modal systems in a purely abstract setting, and potentially could be applied to different concrete scenarios in a systematical way. Take the most recent development of the logic of (epistemic) dependence based on topological space described in[6-7] for instance. There, the language is extended that, not only we have modal operators corresponding to the topological structures of the semantic model, or in other words, the informational representative of each agent, but we can also combine arbitrary groups of agents and compare their epistemic strength in terms of dependence atoms. These linguistic features, as argued in[6], have a natural topological interpretation. However, if we do have located the common abstract structure that supports such extension, and this is shared uniformly by the common mathematical theory we find for all the collection of models within the landscape of information, then through this abstract framework we can introduce similar notions in other logical systems with different semantic models, which greatly helps the design of logic. We will many such development in Chapter 3.

1.2.2 Vertical Comparisons: Transformation between Different Levels

Locating the common structures behind these different classes of models is also of great help for us to find connections among different levels in the landscape. Saying that upper levels corresponds to coarser modelling of semantic information is really a very coarse statement in itself. A much more refined answer should explicitly specify what sort of information within the lower levels are preserved even if we move up, and what type of information is genuinely new, and only present in lower levels.

This type of problems is easier to answer between some levels. For example, it is very intuitive to see that, the difference in terms of informational content between bare epistemic models and plausibility models is the additional plausibility relation possessed by the latter. However, such problem is not so trivial among other levels. Generally speaking, the way to answer this problem is to look at what sorts of *transformations* we could have between these different levels, which brings us to our next problem:

Problem 1.2: What are the possible transformations between different levels of informational structures appearing in the landscape? ◀

In the above Figure 1.1, the model transformations are indicated by the vertical arrows. There are usually two types of transformations we could ask for. Since upper levels are more coarse, usually they are represented by certain special objects in lower levels, through an *embedding*. On the other way around, we could also *(co)freely generate* a model in upper levels using one in lower levels, either by forgetting some structure, or by taking certain closure or substructure. These embed-generate transformation pairs then allow us to arrive at very detailed answers of what sorts of information is present at which level of the landscape of information.

More conceptually, Problem 1.2 sets out to further study *the structure of the landscape itself*, rather than the structure of its individual elements that could serve as classes of models supporting the interpretation of different extensions of modal logic. This in some sense, is a task at another level of abstraction and generality.

From a logical perspective, such model transformations are also naturally accompanied by the question of whether the semantic meaning of certain type of modal languages is *invariant* under such changes. Hence, a complete answer to Problem 1.1 would consist of both ways of describing model changes, and how it is related to modal languages. This would be the main content of Chapter 4.

1.2.3 Horizontal Arrow: Logical Dynamics

Now besides the vertical connections, we could also look at horizontal levels within each layer. For example, in each layer described in Figure 1.1, there are actually infinitely many different modals of the same type, that represents different concrete situations. Furthermore, these models are connected via certain links that, intuitively speaking, describe how two models are related to each other. In more familiar terms to logicians, these links generally correspond to *dynamics* of models, i.e. how the information structure changes under various circumstances.

Now again, such dynamics on the semantic side has a counterpart in the language of modal logic. For different types of dynamics we have different dynamic operators that could be added into the logic systems. Perhaps the most commonly known dynamic logic is PAL, public announcement logic, in the usual relational framework of basic epistemic logics and plausibility models[12-13]. The more general approach of describing model updates and belief changes is presented in DEL, dynamic epistemic logic[14-16]. A more general study of relational upgrades can also be found in[17-18]. A nice survey of these various dynamic logics could be found in[19]. In the setting of evidence-based knowledge and beliefs, we also have corresponding dynamic notions among evidence models[20].

Through a structural perspective, a natural associated problem then is to give a uniform treatment of the different types of dynamics in an abstract setting specified by our answer to Problem 1.1. Different dynamic operators on different levels may appear different at first sight, but in terms of the structural properties they possess, they might indeed play the same role in different contexts. And our results about the correspondence between the mathematical structures of the elements within the landscape on one hand, and the linguistic features on the other hand, could now be extended to the dynamic case. More explicitly, we would also want an answer for the following problem:

Problem 1.3: Is there a structural description of general dynamics of information that universally applies to different levels? If there is, how would such mathematical structure of dynamics correspond to the linguistic feature of general dynamic logic? ◀

In some sense, Problem 1.3 could also be thought as a close relative to Problem 1.1, with the only difference that we are now focus on the reasoning of the dynamics of logic.

This will be our main topic in Chapter 5.

1.3 Approaching the Problems: A Categorical Perspective

In this section, we will explain what is the major choice of the mathematical language we are going to use to formulate, formalise, and analyse the above raised problems, and why we make such a choice. Of course, any ultimate judgement of such a choice should solely rely on how successful the resulting work turns out to be, in terms of addressing the above raised questions. Elegance, simplicity, clearness, and mathematical rigour are all measurements of such success. However, it would also be nice to provide some philosophical justification in advance, so that the readers might be more convinced that they will at least get something out of reading this rather length work.

As indicated in the title of this section, the main mathematical language that will used by us is *category theory*. Category theory was first initiated by two mathematicians, Eilenberg and Mac Lane, in their seminal work[21] in 1945. Their motivation back then was mainly to develop a more conceptualised and clean mathematical language to describe certain methods and results in algebraic topology and algebraic geometry. However, due to its surprisingly great power of abstraction, unification, and generalisation, category theory itself has been vastly studied in the past 70 years or so, and has found its application in almost every branch of mathematics, and also in computer science and logic. The joke is that, category theory is almost like an octopus, with each limb crawling its way into one corner of mathematics and stirring the water up. It is now so deeply integrated into mathematics, in that it has even been suggested to replace set theory as the foundation of mathematics.

Category theory is particularly useful for us, in that it is the best known mathematical framework to study the general and abstract *structure* behind certain mathematical objects. The most frequently mentioned motto of category theory is that, it tends to study the properties of mathematical objects in terms of its *relations with other objects*, other than through its internal structures. In other words, the categorical perspective of mathematics is that, the most important thing is the *structure* of an object, revealed by interacting with other objects, rather than its concrete composition. In fact, *mathematical structuralism* is a new philosophical approach to the nature of mathematics, based on the recent development of category theory and type theory; see[22-23].

To use category theory as a mathematical framework to investigate the information landscape has already been suggested in[24]. Some work has also been done in the general connection and extension of the more familiar relational modal logic and category theory in the thesis[25]. For us, the use of category theory is, in some sense, even more serious, integrated, and essential. From the perspective of the underlying philosophical ideas, the above mentioned general features of category theory are perfectly in line with our *structural* study of the landscape of information. On a more technical level, the connection between our goals as explained in the previous four problems, and the use fo category theory, is at least twofold.

Firstly, on a more micro level, each layer in the landscape of information we would like to study, or in other words, each type of semantics modelling different informational content, can naturally be seen as individual categories. Take the ones that appear in the above Figure 1.1 for instance. We have a category of general Kripke frames **Kr**, and its full substructures, e.g. the subcategory of preorders **Pre**, and the subcategory of equivalence relations **Eqv**. These are the standard types of relational models that are used in various contexts of classical epistemic logic. One step further we also have the category **Top**, the category of topological spaces. Starting from McKinsey and Tarski[26], there has been a long tradition in using topological spaces to support the interpretation of certain modal and intuitionistic systems. As we've mentioned, more recently in[6-7], the category of topological spaces have been used to develop certain new modal systems, allowing reasoning about epistemic dependence. We also have **Evi**, the category of evidence structures, which is explored in[5], and further by more researchers, to develop evidence based knowledge and belief. Commonly seen models in the literature of modal logic also include **Nb**, the category of arbitrary neighbourhood frames, and its full subcategory **Mon** of monotone neighbourhood frames.① These constitute certain benchmark examples we would like to consider along side our general study of the abstract structure of the information landscape, so as to connect our general approach to the concrete modal logic discourse, perhaps more familiar to the readers.

This fact, that most of all the familiar classes of semantic models of modal logic

① Actually, later in Section 2.1 we will show that **Evi** and **Mon** are equivalent as categories, hence, in terms of the resulting modal logic, there would be no difference between them two.

naturally form various categories, makes it a very natural move to provide an answer to Problem 1.1, and some part of Problem 1.3, by specifying certain appropriate categorical properties, which of course, must be satisfied by all the above mentioned exemplars. The possibility of this could also be partially justified by the massive experience we have drawn in studying and applying category theory in practice, such that almost all mathematical structures and constructions could be naturally and elegantly formulated in categorical terms. Furthermore, in many sense category theory is more aligned with logic, in that many categorical structures have a natural counterpart on the syntactic level[27-29]. This is one of the main theme of categorical logic; a nice survey could be found in[30]. For us, all of the above mentioned aspects will be explained in detail in Chapter 3, and some parts of Chapter 5. We will also explain in more detail in the subsequent Section 1.4, that what categorical properties we would like to put out there to serve as the common mathematical structure of those elements in the landscape of information.

The categorical treatment also naturally extends to logical dynamics, as we will show in Chapter 5. For the horizontal dimension in the landscape, dynamic arrows of model change within each level can also be generally and naturally realised as *morphisms* within a category, which allows for a general categorical study of logical dynamics as well. Such a categorical approach of dynamic modal logic is already taken up by several people: See[31] for a generic categorical approach towards relational modal logic; for a very detailed and illuminating categorical exploration of logical dynamics, see[32]. In particular, viewing logical dynamics as morphisms allows us to study the structural properties of the dynamics, and connect it to certain extensions or generalisations of dynamic operators found in the existing literature, providing a satisfactory answer for parts of Problem 1.3.

Secondly, on a more macro level, category theory also provides the best tool to study the mathematical structure of the landscape as a whole, by appealing to the theory of transformations between categories. The individual elements in the landscape, as we've mentioned above, are already categories themselves. This means that the natural notion of transformation between them, are functors, natural transformations, adjunctionc, etc., which are standard categorical notions. As we will see later in Chapter 4, the pair of arrows in Figure 1.1 between two individual levels of information structures are typical examples of an adjunction, which in particular consists of two specified functors and two specified natural transformations. Then the general theory of adjunctions in category theory makes is very easy to study which type of categorical structures are preserves by certain transformations. Once we have identified the correspondence between the categorical structure on one hand, and the syntactical features on the other hand, such results would then smoothly generate answers to part of Problem 1.2 and Problem 1.3, specifying the interpretation of which type of formulas is invariant under transformation of models.

In the future part of this thesis, we will freely use the basic language of category theory, including the notion of categories, functors, (co)limits, and adjunctions. We refer the readers to[33-34] for standard textbooks about general category theory. We will also record some of the more involved lemmas in category theory in the Appendix.

1.4 Duality as Guiding Principle

Bare category theory, like set theory, could serve as the foundational language of mathematics, hence is perhaps too general for our specific task of studying the structure behind the information landscape. As explained in the previous section, the specific categorical structures we would like to put out there will restrain the types of categories we would like to consider, which would, hopefully, pertain more to informational structures and systems of modal logic we have in mind. This section will provide some philosophical background for our choice of the exact categorical structure we would like to consider in this thesis. Again, the total justification must come from the success of such a discourse, but it would be helpful for the readers to understand our philosophical motivations in making such a choice.

Now let's consider the fixed language $\mathcal{L}$ of basic modal logic. Since its syntax has an algebraic signature, perhaps the most general type of semantics for $\mathcal{L}$ is the algebraic semantics (cf. relevant chapters in[35]). However, the algebraic semantics for modal logic is purely abstract, which makes it hard to understand and use for wide applications. In some sense, the very reason that the usual semantics for modal logic, e.g. the relational one provided by Kripke frames, is useful, is precisely because all of them provide certain descriptions of the interpretation of modalities of a *geometric* kind, which facilitates our intuition. It is certainly true that, for most students at a first contact of modal logic, it would be much easier to reason about the modal system using the Kripke semantics of modal operators, than to use one in an algebraic form, at least when the student is not that experienced in general algebraic manipulation.

However, in a sense which could be formulated precisely, to use these semantics of modal logic of a geometric nature, e.g. Kripke frames, topological spaces, neighbourhood frames, etc., does not really lose anything compared to the algebraic semantics, for their corresponding fragments of modal logic. This suggests that there are certain types of *duality* involved between these categories of geometric nature, and the corresponding algebraic categories. We will show in Chapter 3 that, all the benchmark examples of the collection of models mentioned earlier in Section 1.3 admit such type of dualities between certain categories of algebraic nature.

In fact, the idea of a more general type of duality between algebraic data and geometric data is at the very heart of many important branches of modern mathematics after the development of category theory, including algebraic geometry, algebraic topology, functional analysis, etc.. The combination of algebraic rigour and geometric intuition through duality is a fundamental grounding factor behind many advances in modern mathematics. This dualistic perspective also extends to the general discourse of logic, in the form of the *duality between syntax and semantics* (cf.[27,36-37]), with syntax roughly corresponding to algebra, and semantics roughly corresponding to geometry. This is also a central topic in categorical logic, and many important theorems there are establishing dualities between certain fragments of logic and their category of models.

As for a side note, the dualistic picture provides another justification of the use of category theory, in that category theory simply *is* the language of duality. Even the precise notion of duality requires at least certain categorical ideas to formulate in a precise way.

Hence, the duality principle will always serve as a background in the total discourse of this thesis. More related to the task of this section, the duality principle means that, the categorical structure we would like to investigate for the information landscape that admits a modal-type of reasoning would be axiomatised with the aim of *supporting certain geometrical reasoning*. Of course, it should at least include all the previously mentioned examples of categories of semantics as instances of this general structure.

Another factor that influences our choice of general categorical structure is *concreteness*. For all the mentioned elements in the information landscape including **Kr**, **Top**, **Nb**, etc., their objects are all of the form of a set equipped with additional geometric data. Opposed to this, the most general algebraic approach to modal logic loses this concreteness. We believe, as a first systematic study of the information landscape in a categorical perspective, it would be best to remain as concrete as possible, so as to connect to the existing literature on related researches. A more abstract and more general approach than the current one explored in this thesis to more general types of information structure is actually possible, as we will also comment several times later on. However, we leave such a generalisation for future work.

Fortunately for us, there are actually well-established categorical framework that satisfies these two criteria we have in mind, which means we do not have to craft a brand new theory for our study of the information landscape. These are called *topological categories*, or more precisely, *topological categories over* **Set**, the category of sets and functions. The theory of topological categories was first developed in[38], with the exact same aim as ours to axiomatise certain categories that support geometrical, or as the term suggests, topological, reasoning.

Notice that there is a possible conflict of terminology involved. Topological categories are *not* the same thing as the category of topological spaces **Top**; in fact, **Top** is *an instance* of topological categories. However, the terminology of topological categories is already quite standard in the literature, hence we will follow this tradition. More recent approaches could be found in[39-40].

The most general theory of topological categories as developed in[38] allows us to choose an arbitrary base category **X**, and axiomatise geometrical or topological categories *over* **X**. For us, with the aim of concreteness, this base category will always be **Set**. In the future, whenever we say topological categories, we will always mean topological categories over **Set**, unless otherwise specified.

We will first describe the mathematical theory of topological categories in Chapter 2, and use it in later chapters to provide answers to the main problems of this thesis we introduced in Section 1.2, as we have explained in previous texts. The full development and analysis of the four problems in these later chapters will ultimately determine the success of our general choice of this framework.

1.5 Relation to Other Works

As we have mentioned previously in this chapter, there are several previous works adopting a categorical approach of modal logic[31], and dynamic modal logics more generally[25,32]. These works have greatly inspired the current project, but our approach is also significantly different from all of them. To the best knowledge of the author, this thesis is the very first attempt in the literature to use a categorical framework to provide a general account for most of the modal systems, both the basic modal logic and its various static and dynamic extensions, that are widely known in the logical community.

Chapter 2 The Mathematical Theory of Topological Categories

In this chapter, we will provide a technical introduction to the main mathematical framework, viz. topological categories, that will be extensively used in this thesis later when we deal with the problems mentioned in Chapter 1. As we've mentioned, the framework of topological categories over **Set** is introduced in[38] to describe those categories whose objects can be viewed as sets with some additional geometric structures. As motivated in Chapter 1, all the category of models for the epistemic-doxastic modal systems within our interest would be concrete instances of topological categories. The theory of topological categories will use category theory as a basic language.

2.1 Concrete Categories and Concrete Functors

It should be intuitively clear that there will be two main ingredients involved to study geometrical structures over some sets: First, we need to describe arbitrary structures over sets; and second, we need to study when such additional structures are geometrical. This section deals with the first problem.

Let **Top** be the category of topological spaces with continuous functions; it is a typical example of categories with objects being sets equipped with some additional structures. We also have more algebraic examples, such as **Grp**, the category of groups and group homomorphisms. If we regard them as purely *abstract* categories, some valuable information concerning the underlying sets of topological spaces or groups would be lost. The categorical way to retain such information is to consider the *forgetful functor* from **Top** or **Grp** to **Set**, the category of sets and functions. The forgetful functors send each object to its underlying set and each morphism to its underlying set map.

The general theory of concrete categories introduced in[38] actually allows us to describe those categories whose objects are things with additional structure over an *arbitrary base category* **X**, and most of the results in this chapter works at this more general setting. However, for our purpose, we will only consider concrete categories over **Set**. We can generalise the above examples of **Top** and **Grp** to the following definition:

Definition 2.1 (Concrete Category): A *concrete category*, or a *construct*, is a tuple $(\mathcal{A}, |-|)$, consisting of a category $\mathcal{A}$ with a *faithful* functor over **Set**, $|-| : \mathcal{A} \to \mathbf{Set}$.

The functor $|-|$ is called the *forgetful functor*, or the *underlying functor*, of $\mathcal{A}$. When the forgetful functor involved is clear from context, we will also refer $\mathcal{A}$ as a concrete category or a construct, without mentioning the underlying functor explicitly.

The requirement of the forgetful functor $|-|$ being faithful makes sure that the category $\mathcal{A}$ behaves abstractly like a category with objects being sets with additional structures and with morphisms being structure-preserving maps.

Example 2.1 (Concrete Categories): Here we describe several examples of constructs in some detail, which will be relevant to us later:

(i). Trivially, the tuple $(\mathbf{Set}, 1_{\mathbf{Set}})$ with the identity functor on **Set** is a construct.

(ii). Let **Kr** denote the category of Kripke frames. Explicitly, objects in **Kr** are sets equipped with a binary relation, and morphisms in **Kr** are monotone maps between two binary relations.① There is an evident forgetful functor from **Kr** to **Set** that sends each binary relation to its underlying sets.

(iii). More examples can be built from (ii). There is a category **Pre** or preorders with monotone maps, and **Eqv** of equivalence relations with monotone maps. Both of which are full subcategories of **Kr**, and there are evidently induced forgetful functors from **Pre** and **Eqv** to **Set**.

(iv). As we've already mentioned, **Top**, the category of topological spaces, is another major example. The evident forgetful functor sends every topological space to the set of underlying points.

(v). The most general semantic framework of basic modal logic is given by the category **Nb** of *neighbourhood frames*. Its objects are sets X equipped with an arbitrary relation $E \subseteq X \times \wp(X)$, where $\wp(X)$ denotes the power set of X. A morphism between two neighbourhood frames $f : (X, E) \to (Y, F)$ is a function $f : X \to Y$, such that for any $x \in X$ and any $S \subseteq Y$, $fxFS$ implies that $xEf^{-1}(S)$. Again, there is an evident forgetful functor from **Nb** to **Set**, sending each neighbourhood frame to its underlying set.

① This category is referred to as **Rel** in[38], but this name in the current literature on category theory denotes something else, viz. the category with sets as objects and relations between sets as morphisms. This is the convention we will adopt in this paper. *Caveat lector!*

(vi). We could again look at certain subclasses of neighbourhood frames, just like the case for **Pre**, **Eqv** as subclasses of **Kr**. A particularly important full subcategory is **Mon**, the category of *monotone neighbourhood frames*. It is defined as the full subcategory of **Nb**, consisting of neighbourhood frames (X, E) such that for any x, $E[x]$ is upward closed, i.e. xES implies xET for any $S \subseteq T$.

◀

Since for any concrete category $(\mathcal{A}, |-|)$ the forgetful functor is faithful, we will intuitively view hom-sets $\mathcal{A}(A, B)$ for any A, B in $\mathcal{A}$ as subsets of $\mathbf{Set}(|A|, |B|)$, the set of functions from $|A|$ to $|B|$. And for any function $g : |A| \to |B|$, we say it is an $\mathcal{A}$-*morphism* if there exists a morphism $f : A \to B$ in $\mathcal{A}$ such that $|f| = g$. For example, given a function $g : |T| \to |S|$ between the underlying set of two topological spaces, it being a **Top**-morphism exactly means that it is a continuous function $g : T \to S$. Similarly, given a function $g : |X| \to |Y|$ between the underlying sets of two relations, it is a **Kr**-morphism iff it is monotone.

Be careful that, when saying an underlying set map $f : |A| \to |B|$ is an $\mathcal{A}$-morphism, it is crucial to specify out the domain and codomain as the underlying sets of which specific $\mathcal{A}$-*objects*. For example, the same set map $g : |T| \to |S|$ can be either continuous or not, when the bare sets $|T|, |S|$ are equipped with different topologies.

Following the general philosophy of category theory, we also describe what are the corresponding notion of transformation between concrete categories:

Definition 2.2 (Concrete Functor): Given two concrete categories $(\mathcal{A}, |-|_{\mathcal{A}})$ and $(\mathcal{B}, |-|_{\mathcal{B}})$, a *concrete functor* between them is a functor $F : \mathcal{A} \to \mathcal{B}$ over **Set**, i.e. a functor that makes the following diagramme commute,

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad F \quad} & \mathcal{B} \\ & {\scriptstyle |-|_{\mathcal{A}}} \searrow \quad \swarrow {\scriptstyle |-|_{\mathcal{B}}} & \\ & \mathbf{Set} & \end{array}$$

Here we prove a useful technical lemma:

Lemma 2.1: A concrete functor $F : (\mathcal{A}, |-|_{\mathcal{A}}) \to (\mathcal{B}, |-|_{\mathcal{B}})$ is completely determined by its effects on objects.

Proof Suppose F is simply a function on objects from $\mathcal{A}$ to $\mathcal{B}$ that preserves the underlying set. For any objects A, B in $\mathcal{A}$ and any morphism $f : A \to B$, if F can be extend to a concrete functor then the following set map

$$|f| : |FA|_{\mathcal{B}} \to |FB|_{\mathcal{B}},$$

must also be a $\mathcal{B}$-morphism, and if it is, the action of F on f is then uniquely determined by this $\mathcal{B}$-morphism from FA to FB above $|f|$. ∎

Concrete categories and concrete functors also ensemble themselves into a category. We will refer to this category as $\mathfrak{Conc}$. In other words, $\mathfrak{Conc}$ is the full subcategory of the slice category **CAT/Set**, consisting of faithful functors. In fact, $\mathfrak{Conc}$ is even a 2-category in the sense of[41], since there is also a notion of natural transformation between functors, and it naturally relates to the 2-slice of **CAT** over **Set**. However, we will not require such a perspective throughout this thesis.

Remark 2.1: At this point, let us comment on our terminology here. Whenever we use the adjective "concrete" to describe something, in general this will denote those things that are further required to interact well with respect to *all* the forgetful functors involved in that notion. For example, in future places we will use concrete (co)limits to denote those limits that are preserved and lifted by the forgetful functor. There are further such examples, and we will also comment on this along the way. ◀

Example 2.2 (Concrete Functors): Here again we present some examples of concrete functors:

(i). For any concrete category $(\mathcal{A}, |-|)$, there is a uniquely determined concrete functor from $(\mathcal{A}, |-|)$ to $(\mathbf{Set}, 1_{\mathbf{Set}})$, viz. the functor $|-|$ itself. This means that $(\mathbf{Set}, 1_{\mathbf{Set}})$ is the terminal object in the category $\mathfrak{Conc}$.

(ii). The chain of full subcategory inclusions $\mathbf{Eqv} \hookrightarrow \mathbf{Pre} \hookrightarrow \mathbf{Kr}$ are obviously examples of concrete functors.

(iii). Similarly, we have a full subcategory inclusion $\mathbf{Mon} \hookrightarrow \mathbf{Nb}$, which is also a concrete functor.

(iv). A less trivial example is a well-known concrete functor from **Top** to **Pre**. Given any topological space T with topology τ, we can associate it with a preorder on its underlying set $|T|$, called its *specialisation order*, as follows: For any $x, y \in T$

we define

$$x \preceq y \Leftrightarrow \forall U \in \tau[x \in U \Rightarrow y \in U].$$

It is easy to see that this is a well-defined preorder, and we denote it as Spec T. Furthermore, it is again straight forward to verify that continuous functions $f : T \to W$ would induce monotone maps $f : \mathrm{Spec}\, T \to \mathrm{Spec}\, W$. Hence, we have a well-defined concrete functor

$$\mathrm{Spec} : \mathbf{Top} \to \mathbf{Pre}.$$

(v). There is a less known, but equally natural, concrete functor from **Top** to **Mon**, sending each topological space to an associated monotone neighbourhood frame. Explicitly, for any topological space T with topology τ, we define the associated neighbourhood relation E_T as follows: For any $x \in T$ and $S \subseteq T$,

$$xE_T S \Leftrightarrow \exists U \in \tau[x \in U \,\&\, U \subseteq S].$$

In topology, such subsets are called *neighbourhoods* of x. E_T is obviously monotone, and we use N_T to denote the associated neighbourhood frame $(|T|, E_T)$. For more on this construction, see[42]. Now it is easy to see from the definition of morphisms in **Nb** in Example 2.1 (v)., that we have a well-defined functor

$$N_{\_} : \mathbf{Top} \to \mathbf{Mon}.$$

◀

Remark 2.2: In the paper[5], evidence logic is introduced based on neighbourhood style of modal semantics. However, its notion of semantics is slightly different from the usual one we consider for neighbourhood frames. Here we simply summarise it by defining another different category **Evi** of *evidence spaces*, and look at its formal properties. The objects of **Evi** are the same as arbitrary evidence frames (X, E), but it has another notion of morphisms.① A function $f : X \to Y$ consists of a morphism from (X, E) to (Y, F) in **Evi**, iff for any $x \in X$ and any $T \subseteq Y$, $fxFT$ implies that there exists a subset S of X, that xES and $S \subseteq f^{-1}(T)$. Such a definition makes it closely related to the category **Mon** of monotone neighbourhood frames:

① In fact, the notion of evidence spaces we consider here is slightly more general than the one considered in[5]. There they further require that an evidence space (X, E) should satisfy that xEX, and *not* $xE\varnothing$, for any $x \in X$.

Proposition 2.1: **Evi** and **Mon** are concretely equivalent as two concrete categories.

Proof We construct a natural concrete functor from **Evi** to **Mon**. For each evidence space (X, E), we naturally associate it with a monotone neighbourhood frame (X, E^*), by defining for any $x \in X, S \subseteq X$,

$$xE^*S \Leftrightarrow \exists T \subseteq X[xET \,\&\, T \subseteq S].$$

Obviously, (X, E^*) is monotone, and if E is already monotone, then $E = E^*$. This means that if this is a well-defined functor, then it is surjective.

Now we show it is a well-defined functor. For any morphism $f : (X, E) \to (Y, F)$ between two evidence spaces, we claim that $f : (X, E^*) \to (Y, F^*)$ must also be a morphism between neighbourhood frames. Now suppose for some $x \in X$ and $T \subseteq Y$, fxF^*T, which means there exists $S \subseteq T$ that $fxFS$. Since f is a morphism between the two evidence spaces, there exists $U \subseteq X$ that xEU and $U \subseteq f^{-1}T \subseteq f^{-1}S$. This implies that xE^*S, hence f is also a morphism between the two monotone neighbourhood frames.

On the other hand, suppose that $f : (X, E^*) \to (Y, F^*)$ is a morphism between the two associated neighbourhood frames, we show that $f : (X, E) \to (Y, F)$ would also be a morphism between the two evidence spaces. Suppose for some $x \in X$ and $T \subseteq Y$ that $fxFT$, then by definition fxF^*T. It follows that $xE^*f^{-1}(T)$, hence there exists $S \subseteq f^{-1}(T)$ that xES, which exactly means that f is a morphism between the original evidence spaces. This fact shows that the functor **Evi** $\to$ **Mon** is also fully faithful. By any adjoint functor theorem, it follows that this functor **Evi** $\to$ **Mon** establishes a concrete equivalence between the two concrete categories. ∎

Proposition 2.1 means that, working with evidence spaces is essentially the same as working with monotone frames. In fact, from a categorical point of view, the objects does not play such an important role; the real different lies in the definition of morphisms in the two categories. If one contemplate on the definition of morphisms in **Evi**, one could see that it treats a neighbourhood frame (X, E) as a *representation* of its monotone closure E^*. We might come back to this point when discussing the semantics of modal logic associated with (monotone) neighbourhood frames in Chapter 3. ◀

We also say a pair of concrete functors $F : (\mathcal{A}, |-|_{\mathcal{A}}) \rightleftarrows (\mathcal{B}, |-|_{\mathcal{B}}) : G$ is a *concrete adjoint pair*, if F, G as functors between $\mathcal{A}$ and $\mathcal{B}$ are adjoint. In fact, it terms out that all the concrete functors described above in Example 2.2 either have a left or a right concrete adjoint. We will give a much more extensive study of concrete functors and concrete adjunctions between our constructs in interest in Chapter 4.

Remark 2.3: Somewhat surprisingly, the category **Kr** of Kripke frames, is in some sense *universal* among those concrete categories. Modulo a set theoretic condition about measurable cardinals, a theorem in[43] shows that any concrete category can be fully embedded into **Kr**! In particular, both **Top**, **Mon** and **Nb** can be realised as a full subcategory of **Kr**. See[43] for more information on such embeddings.① However, it can be easily seen that such an embedding **Top** $\hookrightarrow$ **Kr** that identifies **Top** as a full subcategory of **Kr** cannot be concrete. There is a simple size argument for this: Let $\mathbb{N}$ be the set of natural numbers, then the cardinality of $\mathbf{Top}_{\mathbb{N}}$ is $2^{2^{\aleph_0}}$, while the cardinality of $\mathbf{Kr}_{\mathbb{N}}$ is simply $2^{\aleph_0}$. Hence, any such embedding cannot be concrete, or preserving the underlying sets. ◀

2.2 Fibres of Concrete Categories

For a construct $(\mathcal{A}, |-|)$, another important aspect of it is reflected in its *fibres*. According to our intuitive understanding of a concrete category $(\mathcal{A}, |-|)$, objects in $\mathcal{A}$ are sets equipped with some additional structures. In many cases we encounter in ordinary mathematics, such structures will not be unique for an arbitrary set X. Hence, it would beneficial to describe the structure of the collection of all objects with the same underlying set. These are precisely fibres for a concrete category.

Definition 2.3 (Fibre): For any construct $(\mathcal{A}, |-|)$ and any set X, the fibre of $\mathcal{A}$ over X, denoted as $\mathcal{A}_X$, is a subcategory of $\mathcal{A}$ with

- objects being $A \in \mathcal{A}$ that $|A| = X$;
- morphisms being $f : A \to B$ in $\mathcal{A}$ that $|f| = \mathrm{id}_X$.

Notice that since $|-|$ is faithful, for any A, B in $\mathcal{A}$ that $|A| = |B| = X$, there are at most one morphism from A to B that is mapped to 1_X. This means that $\mathcal{A}_X$ is actually a

① Here in[43] we have yet another notation **Graph** for what we denote as **Kr**. In more recent categorical literature, **Graph** usually denote the category of directed graphs, or two sets E, V equipped with a source and a target map $s, t : E \to V$. In particular, there can be multiple edges with common source and target. This is the natural category that underlies **Cat**, the (large) category of all (small) categories. *Caveat lector!*

(possible large) *preorder*, and we use the generic symbol $\leq$ to denote this order. We say that two objects A, B in the fibre $\mathcal{A}_X$ are *equivalent* if both $A \leq B$ an $B \leq A$. It is easy to see that if A, B are equivalent in $\mathcal{A}_X$, then they are also isomorphic as objects in the category $\mathcal{A}$, because the forgetful functor $|-|$ is faithful.

We adopt the following terminology concerning fibres of a construct $(\mathcal{A}, |-|)$:

- $(\mathcal{A}, |-|)$ is *amnestic* if $\mathcal{A}_X$ is partially ordered.
- $(\mathcal{A}, |-|)$ is *fibre-small* if for any set X, the fibre $\mathcal{A}_X$ is a *set*, not a proper class.
- $(\mathcal{A}, |-|)$ is *fibre-complete* if arbitrary (possibly large) limits and colimits exists in $\mathcal{A}_X$, i.e. $\mathcal{A}_X$ has all joins and meets.①

In particular, it is easy to see that any concrete category is concretely equivalent to an amnestic one②, and in fact all our canonical examples will be amnestic, in the future texts we will generally assume the constructs to be amnestic, unless otherwise stated.

Example 2.3 (Fibres of Concrete Categories): Here we provide an explicit description of fibres for those concrete categories mentioned in Example 2.1.

(i). For the terminal construct $(\mathbf{Set}, 1_{\mathbf{Set}})$, obviously it has trivial fibres, i.e. for any set X, the fibre $\mathbf{Set}_X$ is the poset with a single element, and it is trivially complete.

(ii). For the category $\mathbf{Kr}$ of Kripke frames, given any set X, the fibre $\mathbf{Kr}_X$ is the set $\wp(X \times X)$ of all binary relations on X. Given any such relation R_1, R_2 on X, the following identity function is monotone

$$1_X : (X, R_1) \to (X, R_2),$$

if, and only if, R_1 is included in R_2. This means that the fibre $\mathbf{Kr}_X$ is ordered by set inclusion, and that the fibre $\mathbf{Kr}_X$ is always a complete lattice.

(iii). For the full subcategories $\mathbf{Pre}$ and $\mathbf{Eqv}$ of $\mathbf{Kr}$, we have a similar description in that $\mathbf{Pre}_X$ and $\mathbf{Eqv}_X$ are the posets of preorder relations and equivalence relations on X, respectively, both ordered by inclusion of relations. They are both complete lattices, because arbitrary intersection of preorders (resp. equivalence relations) is again a preorder (resp. an equivalence relation).

(iv). For topological space $\mathbf{Top}$, the fibre $\mathbf{Top}_X$ consists of the set of topologies on X. However, notice that $\mathbf{Top}_X$ by our definition is not ordered by inclusion of

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① If $\mathcal{A}$ is amnestic and fibre-small, this is the same as requiring $\mathcal{A}_X$ being a complete lattice for any set X.

② Since equivalent objects in the same fibre are necessarily isomorphic, you can always take the poset skeleton of a preorder to obtain a concretely equivalent category.

topologies. For any two topology τ_1, τ_2 on X, the following identity function is continuous

$$1_X : (X, \tau_1) \to (X, \tau_2),$$

if, and only if, every open set in τ_2 is also an open set in τ_1. Hence, in our notation, $\tau_1 \leq \tau_2$ iff $\tau_1 \supseteq \tau_2$, which is the reverse of the inclusion order. It is also easy to see that topologies on X are closed under arbitrary intersections. Hence again, $\mathbf{Top}_X$ is a complete lattice.

(v). For neighbourhood frames **Nb**, the fibre $\mathbf{Nb}_X$ is the set of all relations between X and $\wp(X)$, which is simply $\wp(X \times \wp(X))$. By a similar argument in the case of topological spaces, the identity function on X is a **Nb**-morphism from (X, E_1) to (X, E_2) iff $E_1 \supseteq E_2$. This means that $\mathbf{Nb}_X$ is the dual poset $\wp(X \times \wp(X))^\circ$, hence $\mathbf{Nb}_X$ is also a complete lattice.

(vi). The monotone neighbourhood frames **Mon** inherits the fibre structure from **Nb**, just like **Pre**, **Eqv** inherit from **Kr**. What's also similar is the fact that the fibre $\mathbf{Mon}_X$ is also closed under arbitrary intersection as subsets of $X \times \wp(X)$. Hence, $\mathbf{Mon}_X$ is also complete.

◀

In particular, the above discussion shows us that all these concrete categories described in Example 2.1 are amnestic, fibre-small and fibre-complete. We will see later in this chapter that all the above mentioned constructs are instances of *topological categories*, which are a particular kind of concrete categories whose additional structure encode some geometric information; and all topological categories are amnestic, fibre-small and fibre-complete. We will see this in more detail in the following sections.

Another benefit of describing the fibres is that it allows us to construct new concrete categories from old ones. In particular, we can describe the categorical product of two concrete categories in $\mathfrak{Conc}$, the category of all concrete categories:

Definition 2.4 (Cartesian Product): Given two construct $(\mathcal{A}, |-|_{\mathcal{A}})$ and $(\mathcal{B}, |-|_{\mathcal{B}})$, their Cartesian product is another construct $(\mathcal{A} \times \mathcal{B}, |-|_{\mathcal{A},\mathcal{B}})$, defined as follows:

- It has objects as pairs (A, B) with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, such that they are over the same set, viz. $|A|_{\mathcal{A}} = |B|_{\mathcal{B}}$, and we denote this set as $|(A, B)|_{\mathcal{A},\mathcal{B}}$.
- Given any set map $f : X \to Y$ and any (A, B) over X, (A', B') over Y, the

following function is an $\mathcal{A} \times \mathcal{B}$ morphism

$$f : |(A, B)|_{\mathcal{A},\mathcal{B}} \to \left|(A', B')\right|_{\mathcal{A},\mathcal{B}},$$

if, and only if, $f : |A|_{\mathcal{A}} \to \left|A'\right|_{\mathcal{A}}$ is an $\mathcal{A}$-morphism and $f : |B|_{\mathcal{B}} \to \left|B'\right|_{\mathcal{B}}$ is an $\mathcal{B}$-morphism.

It is then obvious that the operation $|-|_{\mathcal{A},\mathcal{B}}$ extends to a faithful functor, hence $\mathcal{A} \times \mathcal{B}$ is a well-defined construct. Intuitively, objects in $\mathcal{A} \times \mathcal{B}$ are sets equipped with both an $\mathcal{A}$-structure and a $\mathcal{B}$-structure, and this allows us to easily combine different types of structures and form a new category. This point also reflects in the description of fibres of $\mathcal{A} \times \mathcal{B}$, since obviously we have

$$(\mathcal{A} \times \mathcal{B})_X = \mathcal{A}_X \times \mathcal{B}_X.$$

The later is the categorical product in **Pos**, the category of all posets. There are evident projection functors

$$\pi_{\mathcal{A}} : \mathcal{A} \times \mathcal{B} \to \mathcal{A}, \quad \pi_{\mathcal{B}} : \mathcal{A} \times \mathcal{B} \to \mathcal{B},$$

which forgets either the $\mathcal{A}$-structure or $\mathcal{B}$-structure. It is straight forward to see that $\mathcal{A}\times\mathcal{B}$ is indeed the categorical product in $\mathfrak{Conc}$, which we omit the proof here. Furthermore, the above description also extends to arbitrary set indexed product, not only binary ones.

Among all the Cartesian products, there is one type that we particularly care about:

Definition 2.5 (Indexed Product): Given any concrete category $(\mathcal{A}, |-|)$ and any indexed set Σ, there is an induced category $\mathcal{A}^{\Sigma}$ by taking the product of $\mathcal{A}$ over itself Σ-times,

$$\mathcal{A}^{\Sigma} = \prod_{a \in \Sigma} \mathcal{A}.$$

Intuitively, objects in $\mathcal{A}^{\Sigma}$ correspond to sets with Σ-indexed $\mathcal{A}$-structures. For any set X, the fibre $\mathcal{A}^{\Sigma}_X$ simply correponds to the Σ-indexed product of $\mathcal{A}_X$ with itself, which behaves coherently with our notation.

This construction exactly corresponds to how we define semantic models for multi-agent modal logics from single agent case, where the indexed set Σ is understood as the set of agents. Take **Kr** for example: Objects in $\mathbf{Kr}^{\Sigma}$ can be intuitively understood as a set X equipped with a binary relation R_a for each agent $a \in \Sigma$. A function $f : X \to X'$ consists of a morphism between such two structures

$$f : (X, \{R_a\}_{a\in\Sigma}) \to (X', \{R'_a\}_{a\in\Sigma}),$$

exactly when f is monotone for each pair of relations R_a, R'_a. Semantics for multi-agent systems based on other types of models are described similarly.

2.3 Sources, Sinks, and Liftings

As a first approximation, the additional requirement of a concrete category $(\mathcal{A}, |-|)$ to be a topological one lies in some "completeness conditions" with respect to the forgetful functor, which is described by sources, sinks, and their liftings. We describe what they mean and what properties they have in this section carefully.

We call a (possibly large) family or morphisms $\{f_i : A \to A_i\}_{i\in I}$ with the same domain a *source*, and A is called the *domain* of the source.

Definition 2.6 (Structured Source and Lifting): If $(\mathcal{A}, |-|)$ is a construct, then a *structured source* is a (possibly large) family of set maps $\{g_i : X \to |A_i|\}_{i\in I}$. A *lift* of such a structured source is an object A in $\mathcal{A}_X$, such that when these set maps are viewed as functions $g_i : |A| \to |A_i|$, they are all $\mathcal{A}$-morphisms.

The reason we call such object A a lift of a structured source $\mathsf{S} = \{f_i : X \to |A_i|\}_{i\in I}$ is that, given such an object, there is necessarily a uniquely determined source $\{g_i : A \to A_i\}$ in $\mathcal{A}$, such that $|g_i| = f_i$ for any $i \in I$. We also say this uniquely determined source in $\mathcal{A}$ *the lift of* S *on* A.

For instance, given an empty structured source with domain X, any object A in $\mathcal{A}_X$ is a lift. Less trivially, for any $\mathcal{A}$-object B and any function $g : X \to |B|$, a lift of g is then an object A in $\mathcal{A}$ that makes $g : |A| \to |B|$ an $\mathcal{A}$-morphism. More concretely, take **Top** for instance. Given a topological space T and a function $g : X \to |T|$ from an arbitrary set X to $|T|$, a lift of this function can then be identified as a topology on X that makes g continuous.

Among the sources in $\mathcal{A}$, some are particularly special:

Definition 2.7 (Initial Source): A source $\{f_i : A \to A_i\}_{i\in I}$ in $\mathcal{A}$ is called *initial*, if for any B in $\mathcal{A}$ and set map $g : |B| \to |A|$, g is an $\mathcal{A}$-morphism iff each composite

$|f_i| \circ g : |B| \to |A_i|$ is.

It is clear from this definition that being an initial source is a property that is *upward closed*, i.e., if a source is initial in $\mathcal{A}$, then any other larger source containing it would also be initial.

Definition 2.8 (Initial Lift of Structured Source): Given a structured source $\mathsf{S} = \{f_i : X \to |A_i|\}_{i \in I}$, we say it has an *initial lift* in $\mathcal{A}$ if there exists an object A in $\mathcal{A}_X$ lifting S, making the lift of S on A being an initial source in $\mathcal{A}$.

It is easy to observe that initial lifts are defined up to equivalence in $\mathcal{A}_X$:

Lemma 2.2: For any structured source $\mathsf{S} = \{f_i : X \to |A_i|\}_{i \in I}$, suppose A in $\mathcal{A}_X$ is an initial lift of S. For any other A' in $\mathcal{A}_X$, it is also an initial lift of S iff it is equivalent to A.

Proof On the one hand, assume A' and A are equivalent in $\mathcal{A}_X$, then there is an isomorphisms $a : A \to A'$ between A, A' that $|a| = 1_X$. Now suppose A is an initial lift of S. For an $\mathcal{A}$-object B and a set map $g : |B| \to |A'|$, suppose all the underlying maps $f_i \circ g : |B| \to |A_i|$ are $\mathcal{A}$-morphisms. Then since A is an initial lift, $g : |B| \to |A|$ is an $\mathcal{A}$-morphism, which means we have $g' : B \to A$ in $\mathcal{A}$ that $|g'| = g$. We are then able to find $a \circ g' : B \to A'$ in $\mathcal{A}$ that $|a \circ g'| = |g'| = g$, which means $g : |B| \to |A'|$ is also an $\mathcal{A}$-morphism. Hence, A' is also an initial lift of S.

On the other hand, suppose A' is also an initial lift of S. Then obviously, the identity function $1_X : |A| \to |A'|$ is an $\mathcal{A}$-morphism, because all the composits $f_i \circ 1_X = f_i : |A| \to |A_i|$ are. This shows $A \leq A'$. A similar argument implies $A' \leq A$, which means A and A' are equivalent in $\mathcal{A}_X$. ∎

For now, let's just look at the initial lifts of the most simple kind of structured sources, viz. the empty ones, as an example. According to the definition, for any set X, the empty structured source with domain X has an initial lift A in $\mathcal{A}_X$, iff for any other $\mathcal{A}$-object B, any set map $f : |B| \to |A|$ is in fact an $\mathcal{A}$-morphism. Such objects are called *codiscrete* objects in $\mathcal{A}$. We can extend this to the following result:

Lemma 2.3: For any construct $(\mathcal{A}, |-|)$, if it has initial lifts for all the empty structures sources with arbitrary domains, then the forgetful functor $|-|$ has a fully faithful right adjoint

$$\text{codisc} : \mathbf{Set} \to \mathcal{A},$$

which identifies **Set** as a reflexive subcategory of $\mathcal{A}$.

Proof We explicitly construct a concrete functor codisc $: \mathbf{Set} \to \mathcal{A}$, with **Set** being viewed as equipped with the identity functor to **Set** itself. For any set X, we set codisc X to be the initial lift of the empty structured source on X. Since any set map $f : |A| \to |\text{codisc}\, X|$ is an $\mathcal{A}$-morphism, by Lemma 2.1 the action of codisc on morphisms is uniquely determined, and it is indeed a concrete functor. Such properties imply that have an isomorphism between hom-sets

$$\mathcal{A}(A, \text{codisc}\, X) \cong \mathbf{Set}(|A|, X).$$

Its naturality is easy to verify. Hence, we have an adjunction $|-| \dashv$ codisc. From our construction, it is easy to verify that

$$|-| \circ \text{codisc} = 1_{\mathbf{Set}},$$

which implies the counit of this adjunction is identity. Thus, codisc is also a concrete functor, and this adjunction identifies **Set** has a reflexive subcategory of $\mathcal{A}$. ∎

Lemma 2.3 also tells us that having initial lifts for all empty structured sources is a much stronger property than merely requiring that the forgetful functor has a right adjoint. There are many examples of adjunctions with a faithful left adjoint which is not reflexive.

The structure of the fibres $\mathcal{A}_X$ is also closely connected to (initial) lifts of structured sources. For any structured source $\mathsf{S} = \{f_i : X \to |A_i|\}_{i \in I}$, we write $\mathcal{A}_{\mathsf{S}}$ for full subcategory of $\mathcal{A}_X$ consisting of lifts of S. We also write X for the empty source with domain X, and since every object in $\mathcal{A}_X$ is a lift of this empty structured source, our notation for $\mathcal{A}_X$ is consistent. The following result shows that $\mathcal{A}_{\mathsf{S}}$ is a downward closed subset of $\mathcal{A}_X$ for any source S:

Lemma 2.4: Let $(\mathcal{A}, |-|)$ be a construct. For any structured source $\mathsf{S} = \{f_i : X \to |A_i|\}_{i \in I}$ with any set X, if A is a lift of this source and $A' \leq A$ in $\mathcal{A}_X$, then so is A'.

Proof Assume $A' \leq A$ in $\mathcal{A}_X$. This means that there is a unique $\mathcal{A}$-morphism $g : A' \to A$ that $|g| = 1_X$. If A is a lift of S, it follows that there are $\mathcal{A}$-morphisms $g_i : A \to A_i$

that $|g_i| = f_i$. Then we have

$$f_i = |g_i| \circ |g| = |g_i \circ g|,$$

which shows that each f_i viewed as functions $f_i : |A'| \to |A_i|$ is also an $\mathcal{A}$-morphism. This means that A' is also a lift of S. ∎

We can show that, for a structured source $\mathsf{S} = \{f_i : X \to |A_i|\}_{i \in I}$, having an initial lift implies it is actually a principal ideal:

Lemma 2.5: Suppose the structured source $\mathsf{S} = \{f_i : X \to |A_i|\}_{i \in I}$ has an initial lift A. Then the downward closed set $\mathcal{A}_\mathsf{S}$ is *principal*, and we have

$$\mathcal{A}_\mathsf{S} = \{ B \in \mathcal{A}_X \mid B \leq A \}.$$

Proof Suppose $B \in \mathcal{A}_\mathsf{S}$. By definition, for any $i \in I$, $f_i : |B| \to |A_i|$ is an $\mathcal{A}$-morphism. Since A is an initial lift, this means that the identity function $1_X : |B| \to |A|$ is also an $\mathcal{A}$-morphism. Hence, $B \leq A$. Combined with the fact that $\mathcal{A}_\mathsf{S}$ is downward closed, we know that $\mathcal{A}_\mathsf{S}$ is identical to the principal ideal generated by A. ∎

In particular, Lemma 2.5 shows that, if for some class of structured sources the construct $(\mathcal{A}, |-|)$ has initial lifts, then joins of subsets of the form $\mathcal{A}_\mathsf{S}$, where S is a structured source in the class, exists in the fibre $\mathcal{A}_X$, where X is the domain of S. For example, suppose the construct has all initial lifts for empty structured sources. According to our previous discussion, this means that all the fibres $\mathcal{A}_X$ has a top element, which are exactly the codiscrete objects.

What we have described above in this section have a completely dual theory, by interpreting what we have described above in the dual category $\mathcal{A}^{\mathrm{op}}$. In particular, we have a notion *sink* in $\mathcal{A}$ dual to that of a source, which is simply a source in $\mathcal{A}^{\mathrm{op}}$. In elementary terms, a sink in $\mathcal{A}$ is a (possibly large) collection of morphisms $\{f_i : A_i \to A\}_{i \in I}$ with common codomain. Similarly, we have the definition of *structured sinks* and *lifts of structured sinks*. What's worthwhile to fully specify again is the notion of *final sink* and *final lift of structured sinks*, dual to the notion of initial sources and the initial lift of a structured source:

Definition 2.9 (Final Sink): A sink $\{f_i : A_i \to A\}_{i \in I}$ in $\mathcal{A}$ is *final*, if for any B in $\mathcal{A}$ and any set map $g : |A| \to |B|$, g is an $\mathcal{A}$-morphism iff each composite $g \circ |f_i| : |A_i| \to |B|$ is.

Definition 2.10 (Final Lift of Structured Sink): Given a structured sink $\{f_i : |A_i| \to X\}_{i \in I}$, we say it has a *final lift* in $\mathcal{A}$ if there exists an object A in $\mathcal{A}_X$ lifting it, making this lifted sink in $\mathcal{A}$ final.

In particular, a structured sink S of the construct $(\mathcal{A}, |-|)$ is the same as a structured source for $(\mathcal{A}^{\mathrm{op}}, |-|^{\mathrm{op}})$.① This directly implies the following results:

Lemma 2.6: For any structured sink $\mathsf{S} = \{f_i : |A_i| \to X\}_{i \in I}$, suppose A in $\mathcal{A}_X$ is a final lift of S. For any other A' in $\mathcal{A}_X$, it is also a final lift of S iff it is equivalent to A.

Proof Directly apply Lemma 2.2 when we view S as a structured source of $(\mathcal{A}^{\mathrm{op}}, |-|^{\mathrm{op}})$. ∎

The dualisation of Lemma 2.3 would also suggest the following holds:

Lemma 2.7: For any construct $(\mathcal{A}, |-|)$, if it has final lifts for all the empty structures sinks with arbitrary codomains, then the forgetful functor $|-|$ has a fully faithful left adjoint

$$\mathrm{disc} : \mathbf{Set} \to \mathcal{A},$$

which identifies **Set** as a coreflexive subcategory of $\mathcal{A}$.

Proof $(\mathcal{A}, |-|)$ has all final lifts for empty structured sinks means that $(\mathcal{A}^{\mathrm{op}}, |-|^{\mathrm{op}})$ has all initial lifts for empty structured sources. Lemma 2.3 then implies that $|-|^{\mathrm{op}}$ has a fully faithful right adjoint, which, equivalently, suggests that $|-|$ has a fully faithful left adjoint,

$$\mathrm{disc} \dashv |-|.$$

This then identifies **Set** as a coreflexive subcategory of $\mathcal{A}$. ∎

Following down the road, we would not be surprised to find the remaining dualisation results. Given a structured sink S, if we write $\mathcal{A}^{\mathsf{S}}$ as the full subcategory of $\mathcal{A}_X$ consisting of lifts of S, then we have:

Lemma 2.8: Let $(\mathcal{A}, |-|)$ be a construct. For any structured sink S with codomain any set X, $\mathcal{A}^{\mathsf{S}}$ is upward closed.

① For any functor $F : \mathcal{A} \to \mathcal{B}$, it naturally induces a dual functor $F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$. To be fully formal, we haven't defined what is a structured source for a functor of type $F : \mathcal{X} \to \mathbf{Set}^{\mathrm{op}}$. However, it should be clear from the context that all the previous definitions have not used any specific properties of **Set**, hence could be extended to the case where we have a category over *any* based category; and all the lemmas also hold in this full generality.

Proof S is a structured source in $(\mathcal{A}^{\text{op}}, |-|^{\text{op}})$. Lemma 2.4 suggests that $\mathcal{A}_S^{\text{op}}$ is downward closed, which exactly means that $\mathcal{A}^{\mathsf{S}}$ is upward closed. ∎

Finally, we have:

Lemma 2.9: Suppose the structured sink $\mathsf{S} = \{f_i : X \to |A_i|\}_{i \in I}$ has a final lift A. Then the upward closed set $\mathcal{A}^{\mathsf{S}}$ is *principal*, and we have

$$\mathcal{A}^{\mathsf{S}} = \{\, B \in \mathcal{A}_X \mid A \leq B \,\}.$$

Proof It is easy to see from definition that A is a final lift of the structured sink S for $(\mathcal{A}, |-|)$ iff it is the initial lift of S viewed as a structures source for $(\mathcal{A}^{\text{op}}, |-|^{\text{op}})$. The rest follows trivially from Lemma 2.5. ∎

Up till this point, we have completely dualised all the results for sources to sinks. The upshot of this section is the following duality theorem:

Theorem 2.1: For any concrete category $(\mathcal{A}, |-|)$, it has initial lifts for all structured sources if, and only if, it has final lifts for all structured sinks.

Proof Suppose $(\mathcal{A}, |-|)$ has initial lifts for all structured sources. We show it also has final lifts for structured sinks. The other direction follows completely from a dualising argument. Given any structured sink $\mathsf{S} = \{f_i : |A_i| \to X\}_{i \in I}$, we associate it with a structured source $\mathsf{S}_\flat$ as follows,

$$\mathsf{S}_\flat := \{\, g : X \to |B| \mid \forall i[g \circ f_i : |A_i| \to |B| \text{ is an } \mathcal{A}\text{-morphism}] \,\},$$

and let A be the initial lifting of $\mathsf{S}_\flat$. First, notice that for any $f_i : |A_i| \to |A|$, by definition for any $g : X \to |B|$ in $\mathsf{S}_\flat$, $g \circ f_i : |A_i| \to |B|$ is an $\mathcal{A}$-morphism. By the initiality of A w.r.t. $\mathsf{S}_\flat$, this shows that each $f_i : |A_i| \to |A|$ is an $\mathcal{A}$-morphism, which shows A is indeed a lift of S. To show it's the final lift of S, suppose we have a set map $g : |A| \to |B|$ that for all f_i, the composite $g \circ f_i : |A_i| \to |B|$ is an $\mathcal{A}$-morphism. This is exactly saying that $g \in \mathsf{S}_\flat$, hence by the fact that A is a lift of $\mathsf{S}_\flat$ it follows that $g : |A| \to |B|$ is also an $\mathcal{A}$-morphism. This completes the proof that A is the final lift of the structured sink S. ∎

Remark 2.4: In the proof of Theorem 2.1, it is essential that we require the construct $(\mathcal{A}, |-|)$ to have initial or final lifts for possibly *large* structures sources and sinks. If $\mathcal{A}$ is not small, which will indeed be the case for most of our examples in mind, then the induced structured source $\mathsf{S}_\flat$ need not be small, even if we start from a small structured sink S. In fact, in many algebraic cases, for certain natural forgetful functors we have all initial lifts for all small structured sources, but the converse on final lifts does not hold. For more on this, see relevant discussions in[38]. ◀

2.4 Topological Categories

Finally, we have all the relevant background to describe the main subject of this chapter, *topological categories*.

Definition 2.11 (Topological Category): A construct $(\mathcal{A}, |-|)$ is a *topological category* (over **Set**), if every (possibly large) structured source has a *unique* initial lift.

Equivalently, as Theorem 2.1 suggests, a construct $(\mathcal{A}, |-|)$ is a topological category if every (possibly large) structured sink has a unique final lift. The uniqueness involved in Definition 2.11 is a stronger requirement than the mere existence of initial of final lifts. When it is clear in the context which functor is taken to be the forgetful functor, we also refer to $\mathcal{A}$ as a topological category. As a trivial example, **Set** equipped with the identity functor $(\mathbf{Set}, 1_{\mathbf{Set}})$ is obviously a topological category, since every underlying morphism is, by definition, a **Set**-morphism. Definition 2.11 induces $\mathfrak{Topc}$, the category of topological categories, as a subcategory of $\mathfrak{Conc}$, the category of concrete categories, both of which are full subcategories of the slice category **CAT/Set**. We will provide a more detailed study of the categorical structure of these meta-categories in Chapter 4.

Simply from this definition, we can immediately extract many nice properties of topological categories. For example, from Lemma 2.7 and Lemma 2.3, we know that there is both a left and right adjoint of $|-|$,

$$\mathrm{disc} \dashv |-| \dashv \mathrm{codisc},$$

which identifies **Set** as both a reflexive and coreflexive subcategory of $\mathcal{A}$. In particular, $|-|$ preserves both limits and colimits. In fact, more is true:

Lemma 2.10: If $(\mathcal{A}, |-|)$ is a topological category, then $|-|$ also lifts limits and colimits uniquely.

Proof We prove the case for limits. The case for colimits is complete dual. Given any diagramme $D : \mathcal{I} \to \mathcal{A}$, let L be the limit of $|-| \circ D$ in **Set**. Then there is a naturally induced structured source as follows,

$$\{l_i : L \to |D_i|\}_{i \in \mathcal{I}}.$$

Since $\mathcal{A}$ is topological, we have a unique initial lift A of this structured source. We want to show that A is the limit for the diagramme D in $\mathcal{A}$, but this is immediate from the definition of initial lift. ∎

Corollary 2.1: If $(\mathcal{A}, |-|)$ is a topological category, then it is complete and cocomplete, and the forgetful functor $|-|$ preserves them.

Proof This follows from the fact that **Set** is complete and cocomplete, and by Lemma 2.10 $|-|$ lifts them. $|-|$ preserves them because it has both a left and right adjoint. ∎

Intuitively, Lemma 2.10 means that the underlying set of the (co)limits of a digramme in the topological category $\mathcal{A}$ is the same as the (co)limits of the underlying diagramme in **Set**. When this happens, we say the construct has *concrete (co)limits*. In particular, all topological categories have concrete (co)limits. Corollary 2.1 also suggests that every topological category is *fibre-complete*, since meets and joins in a fibre $\mathcal{A}_X$ is simply special type of limits and colimits in $\mathcal{A}$.

The above concerns with limits and colimits within a single topological category. We may also show that the class of topological categories is closed under forming products of topological categories themselves: If both $(\mathcal{A}, |-|_{\mathcal{A}})$ and $(\mathcal{B}, |-|_{\mathcal{B}})$ are topological, then so is the previously constructed product category $\mathcal{A} \times \mathcal{B}$:

Theorem 2.2: If both $(\mathcal{A}, |-|_{\mathcal{A}})$ and $(\mathcal{B}, |-|_{\mathcal{B}})$ are topological categories, then so is their product $\mathcal{A} \times \mathcal{B}$.

Proof The initial or final lifts in $\mathcal{A} \times \mathcal{B}$ are computed point-wise. For any structured source on $\mathcal{A} \times \mathcal{B}$ of the following form

$$\{f_i : X \to |(A_i, B_i)|_{i \in I}\},$$

we can lift it individually in $\mathcal{A}$ and $\mathcal{B}$, i.e., we let A, B be the initial lift of the following induced structured sources on $\mathcal{A}$, $\mathcal{B}$, respectively,

$$\{f_i : X \to |A_i|\}_{i \in I}, \quad \{f_i : X \to |B_i|_{i \in I}\}.$$

We then obtain an object (A, B) of $\mathcal{A} \times \mathcal{B}$. From here it is then straight forward to verify that it is indeed the initial lift of the original structured source. ∎

It is easy to see that the same proof works for arbitrary set indexed products, not only for binary ones. Hence, we have the following corollary:

Corollary 2.2: If $(\mathcal{A}, |-|)$ is a topological category, then so is $(\mathcal{A}^{\Sigma}, |-|^{\Sigma})$ for any indexed set Σ.

Proof By construction $\mathcal{A}^{\Sigma}$ is the product of $\mathcal{A}$ with itself for Σ times, hence Theorem 2.2 applies. ∎

But at this point, we are not able to explicitly verify that our main categories in interest are topological, since it is *a priori* very hard to directly verify that every large structured source has a unique initial lift. However, we can obtain some other equivalent conditions which are easier to verify, when the constructed involved are better behaved. For instance, our main examples of topological categories are all fibre-small, as explicitly described in Example 2.3. For these construct, to test whether they are topological categories, we need not to verify whether the initial or final lift exists for arbitrarily large structured sources or sinks, which is quite scary. The following results tells that for fibre-small constructs, checking the existence of initial or final lifts only for small structured sources or sinks suffices.

Lemma 2.11: If the construct $(\mathcal{A}, |-|)$ is fibre-small, then the following conditions are equivalent:

1. $\mathcal{A}$ is topological;
2. Every *small* structured source has a unique initial lift;
3. Every *small* structured sink has a unique final lift.

Proof Again, we simply show the equivalence of (1) and (2); the equivalence for (3) follows from duality. (1) implies (2) is immediate. Suppose now (2) holds. Given any large structured source $\{f_i : X \to |A_i|\}_{i \in I}$, every single set map $f_i : X \to |A_i|$ has an initial lift, which we denote as $B_i \in \mathcal{A}_X$. Since $\mathcal{A}_X$ is fibre-small, the collection $\{ B_i \mid i \in I \}$ must be a *set*, not a proper class, which means that there exists a *subset* J of I, such that

$$\{ B_i \mid i \in I \} = \{ B_i \mid i \in J \}.$$

Let A be the unique initial lift for this small structured source. In particular, by Lemma 2.5 this suggests that

$$A \leq B_i, \ \forall i \in J.$$

Now for any i in the large class I, we observe that the morphism $f_i : |A| \to |A_i|$ must be an $\mathcal{A}$-morphism. Because by definition, the initial lift of $f_i : X \to |A_i|$ is some B_j for some $j \in J$, hence $f_i : |B_j| \to |A_i|$ is an $\mathcal{A}$-morphism; and we know that $A \leq B_j$ hence $1_X : |A| \to |B_j|$ is also an $\mathcal{A}$-morphism. This means the composite $f_i : |A| \to |A_i|$ is an $\mathcal{A}$-morphism as well. Now we have two sources in $\mathcal{A}$, $\{A \to A_i\}_{i \in J}$ and $\{A \to A_i\}_{i \in I}$, the former of which is initial by definition. But since J is included in I, this automatically implies that the larger source $\{A \to A_i\}_{i \in I}$ is initial. Hence, A is also the unique initial lift for this large structured source. ∎

Lemma 2.11 greatly simplifies things, in that we only need to worry about essentially small things. Since as Examples 2.3 shows, all the examples we care about are fibre-small, we will henceforth assume that all the constructs we state are fibre-small, unless otherwise stated.

However, checking that a construct has initial or final lifts for every small structured source or sink could also be tedious. We want to further simplify the conditions such that we can immediately recognise all the main categories listed in Example 2.1 are indeed topological. To this point, we can recall how we construct the weak topology induced in the domain X when we specify a (small) family of functions $\{f_i : X \to |T_i|\}_{i \in I}$ with each codomain topologised as T_i. We simply take the categorical product of all the codomains $\prod_{i \in I} T_i$, and compute the weak topology induced by the single uniquely induced function $\langle f_i \rangle_{i \in I} : X \to \left|\prod_{i \in I} T_i\right|$. Notice that we can write in this way because products in **Top** has the same underlying set as the usual Cartesian products of sets. With this in mind, we prove the following result.

Theorem 2.3: A construct $(\mathcal{A}, |-|)$ with small fibres is topological iff the following conditions hold:

- $|-|$ preserves and lifts *small products*, i.e. $\mathcal{A}$ has small concrete products;
- $\mathcal{A}$ has all unique initial lifts for *structured injectives*, viz. structured sources consisting of injective functions of the form $S \hookrightarrow |A|$;

- $|-|$ has a concrete right adjoint, viz. a concrete functor

$$\operatorname{codisc} : \mathbf{Set} \to \mathcal{A},$$

such that $|-| \dashv \operatorname{codisc}$.

Proof If $(\mathcal{A}, |-|)$ is topological then obviously all the conditions holds. Hence, we only prove the converse. From Lemma 2.11 we know that we only need to check the existence of initial lifts for small structured sources. Suppose we are given $\{f_i : X \to |A_i|\}_{i\in I}$ where I is small. Since $|-|$ preserves and lifts all small products, we can construct the product $\prod_{i\in I} A_i$ in $\mathcal{A}$, and we must have

$$\left|\prod_{i\in I} A_i\right| = \prod_{i\in I} |A_i|,$$

where on the right hand side the product is taken in **Set**. Hence, there is an induced structured source of the form

$$\langle\langle f_i\rangle_{i\in I}, 1_X\rangle : X \to \left|\prod_{i\in I} A_i\right| \times X = \left|\prod_{i\in I} A_i \times \operatorname{codisc} X\right|.$$

The equality uses the fact that the induced codiscrete object $\operatorname{codisc} X$ has the underlying set X. This function is obviously injective, hence we have a unique initial lifting A. We claim that this is indeed the initial lift for the original source, which follows from the formal properties of products. First of all, we have a uniquely determined $\mathcal{A}$-morphism

$$\langle\langle g_i\rangle_{i\in I}, g\rangle : A \to \prod_{i\in I} A_i \times \operatorname{codisc} X,$$

whose underlying function is $\langle\langle f_i\rangle_{i\in I}, 1_X\rangle$. By the defining property of codiscrete object, $g : A \to \operatorname{codisc} X$ is uniquely determined by the underlying function $1_X : |A| \to X$, hence it contains no true new information. Obviously, the source $\{g_i : A \to A_i\}_{i\in I}$ is a lift of the original structured source $\{f_i : X \to |A_i|\}$. For any $f : |B| \to |A|$ that $f_i \circ f : |B| \to |A_i|$ are all $\mathcal{A}$-morphisms, and suppose $|h_i| = f_i \circ f$, then so is the following function

$$\langle\langle f_i \circ f\rangle_{i\in I}, f\rangle : |B| \to \left|\prod_{i\in I} A_i \times \operatorname{codisc} X\right|.$$

The $\mathcal{A}$-morphism above it is the following map determined by the product and codiscrete structure,

$$\langle \langle h_i \rangle_{i \in I}, f' \rangle \,:\, B \to \prod_{i \in I} A_i \times \operatorname{codisc} X,$$

where f' is uniquely determined by f since codisc X is codiscrete. Then by initiality of A, we can derive f is also an $\mathcal{A}$-morphism, and hence A is also the initial lift for the original structured source. ∎

In the above proof, we have used the existence of products to combine a structured source of cardinality lager than 1 to a single structured morphism. The existence of codiscrete object then allows us to conveniently transform this to an injective morphism. After that, the existence of liftings of injective structured morphisms then does the remaining work. The conditions listed in Theorem 2.3 are much more easy to check for our examples.

Corollary 2.3: For any indexed set Σ, $\mathbf{Kr}^\Sigma$, $\mathbf{Pre}^\Sigma$, $\mathbf{Eqv}^\Sigma$, $\mathbf{Top}^\Sigma$, $\mathbf{Nb}^\Sigma$ and $\mathbf{Mon}^\Sigma$, are all topological categories.

Proof We first focus on the case when Σ is the singleton set. It is well-known that all the categories **Kr**, **Pre**, **Eqv**, **Top**, **Nb** and **Mon** have concrete products and codiscrete structures. In all these categories there is also a canonical notion of a structure restricted to a subset: In **Kr** this is simply given by relations restricted to the subset; in **Top** we have the subspace topology; similarly we have the restricted evidence relation on a subset in **Mon** and **Nb**. These provide the initial lifts for structured sources which are injective morphisms. Then by Theorem 2.3 we know that they are topological categories. Theorem 2.2 implies the desired result for arbitrary indexed set Σ. ∎

Remark 2.5: Let's further consider the category **Evi** of evidence spaces we have introduced before. In Remark 2.2, we have shown that there is a concrete equivalence between **Evi** and **Mon**, which means **Evi** is almost as good as a topological category. However, according to Definition 2.11, **Evi** is not topological. The only obstruction is that, the fibres $\mathbf{Evi}_X$ for any set X is not a lattice, but only a complete and cocomplete preordered set. In other words, there could be non-identical evidence spaces (X, E) and (X, F), based on the same set X, which are isomorphic in **Evi**. In a precise sense, the category **Mon** of monotone neighbourhood frames is precisely the category modulo the congruence generated by such isomorphisms. However, it is also possible to work with a more general definition of topological categories that does include examples like **Evi**. In some sense, this alternative approach would be more natural in a categorical perspective, since it would then be invariant in terms of concrete equivalence. However, such a treatment is beyond the scope of this thesis, and we are happy to work with **Mon** as a nice substitute of topological category for **Evi**. ◀

By Lemma 2.5, we know that given a structured source $\mathsf{S} = \{f_i : X \to |A_i|\}_{i \in I}$ in $(\mathcal{A}, |-|)$, if $\mathcal{A}$ is indeed topological, then $\mathcal{A}_{\mathsf{S}}$ is a principal ideal, and the initial lift for S must be the top element of $\mathcal{A}_{\mathsf{S}}$. Now if we write $\mathcal{A}_{f_i}$ for the lifts of the single structured source $\{f_i : X \to |A_i|\}$, then in general, we must have

$$\bigvee \mathcal{A}_{\mathsf{S}} = \bigwedge_{i \in I} \mathcal{A}_{f_i},$$

where the meet and join are taken in the complete lattice $\mathcal{A}_X$. There is an equivalent description of the initial lift according to the proof of Theorem 2.3. We can first take the categorical product of all the A_i, and obtain the following single structured source:

$$\langle f_i \rangle_{i \in I} : X \to \left| \prod_{i \in I} A_i \right|.$$

Then, the initial lift on X of the original structured source is exactly the same as the initial lift of this single function.

There is also a completely dual description for final lifts. Given a structured sink $\mathsf{S} = \{f_i : |A_i| \to X\}_{i \in I}$, $\mathcal{A}^{\mathsf{S}}$ would be a principal filter, and the final lift of S is the bottom element of $\mathcal{A}^{\mathsf{S}}$. Similarly, we must have

$$\bigwedge \mathcal{A}^{\mathsf{S}} = \bigvee_{i \in I} \mathcal{A}^{f_i}.$$

2.5 Topological Categories as Bifibrations

In this section we provide an alternative perspective on topological categories, which emphasis on the structure of the fibres of a topological category. Given any set map $f : X \to Y$, we can construct a map between the fibres:

$$f^* : \mathcal{A}_Y \to \mathcal{A}_X.$$

For any B in $\mathcal{A}_Y$, we define f^*B to be the unique initial lift of the structured source

$$f : X \to |B|.$$

Similarly, we can also define a direct image map between the fibres:

$$f_! : \mathcal{A}_X \to \mathcal{A}_Y.$$

For any A in $\mathcal{A}_X$, we define $f_!A$ to be the unique final lift of the structured sink

$$f : |A| \to Y.$$

From our definition, it is then easy to see that given $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, we have the following equivalence,

$$A \leq f^*B \Leftrightarrow f : |A| \to |B| \text{ is an } \mathcal{A}\text{-morphism } \Leftrightarrow f_!A \leq B.$$

In particular, this shows we have an adjunction between fibres $\mathcal{A}_X$ and $\mathcal{A}_Y$,

$$f_! \dashv f^*.$$

In particular, we will call $f_!$ the *push-forward* map, and f^* the *pullback map*. Since they are maps between different fibres in $\mathcal{A}$ over different sets, we call them *fibre-connections*. The first observation is that, they are both functorial in the following sense:

Lemma 2.12: Let $(\mathcal{A}, |-|)$ be a topological category. Given functions $f : X \to Y$ and $g : Y \to Z$ between sets, the following holds:

$$f^* \circ g^* = (gf)^*, \quad g_! \circ f_! = (gf)_!.$$

Proof We show pullback maps are functorial, and the case for push-forward automatically follows from the fact that it is the left adjoint of pullbacks. Suppose we have an object A in the fibre $\mathcal{A}_Z$. To show that $f^* \circ g^* = (gf)^*$, we only need to show that the pullback f^*g^*A is indeed the initial lift of gf. Given any function $h : |B| \to X$, by definition, it is an $\mathcal{A}$-morphism when viewed as a map $h : |B| \to \left|f^*g^*A\right|$ iff $fh : |B| \to \left|g^*A\right|$ is an $\mathcal{A}$-morphism, since f^*g^*A is the initial lift of g^*A along f; similar argument shows that $fh : |B| \to \left|g^*A\right|$ is an $\mathcal{A}$-morphism iff $gfh : |B| \to |A|$ is an $\mathcal{A}$-morphism, which indeed shows that f^*g^*A is the initial lift of A along gf. ∎

These maps completely determines the initial and final lifts of any structured source or sink in a topological category:

Lemma 2.13: Suppose $(\mathcal{A}, |-|)$ is a topological category, then for any structured source $\{f_i : X \to |A_i|\}_{i\in I}$, its initial lift is given by $\bigwedge_{i\in I} f_i^* A_i$. Similarly, for any structured sink $\{f_i : |A_i| \to X\}_{i\in I}$, its final lift is given by $\bigvee_{i\in I}(f_i)_! A_i$.

Proof We only prove the case for initial lift; the other case is completely similar. First of all, since we know that

$$\bigwedge_{i\in I} f_i^* A_i \leq f_i^* A_i,$$

it follows that $f_i : \left|\bigwedge_{i\in I} f_i^* A_i\right| \to |A_i|$ is an $\mathcal{A}$-morphism, for any $i \in I$. On the other hand, for any $g : |B| \to \left|\bigwedge_{i\in I} f_i^* A_i\right|$, it is an $\mathcal{A}$-morphism iff the following holds,

$$B \leq g^* \bigwedge_{i\in I} f_i^* A_i = \bigwedge_{i\in I} g^* f_i^* A_i = \bigwedge_{i\in I} (f_i \circ g)^* A_i,$$

iff for any $i \in I$, $B \leq (f_i \circ g)^* A_i$, i.e. $f_i \circ g : |B| \to |A_i|$ is an $\mathcal{A}$-morphism. The above equality uses the fact that g^* is a right adjoint, hence preserves arbitrary intersections, and that pullback maps are functorial. Hence, this shows that $\bigwedge_{i\in I} f_i^* A_i$ is indeed the initial lift of the structure source we start with. ∎

Lemma 2.12 actually shows that a topological category induces two functors,

$$\mathbf{Set} \to \mathbf{SupL}, \quad \mathbf{Set}^{\mathrm{op}} \to \mathbf{InfL},$$

where **SupL** is the category of complete lattices with morphisms being maps preserving arbitrary joins, and **InfL** is the category with the same objects but with morphisms being those maps that preserve arbitrary meets. Usually, objects in **SupL** are called *suplattices*, while objects in **InfL** are called *inflattices*, though in fact, they are the same class of objects, but with different notions of morphisms.

In fact, each of the above functor determines the other one. We can see this either from the micro level, that any join-preserving map between complete lattices has a right adjoint, and similarly each meet-preserving map has a left adjoint. Or from the macro level, it is well-known that $\mathbf{SupL} \cong \mathbf{InfL}^{\mathrm{op}}$, hence any functor from **Set** to **SupL** is in fact a functor from **Set** to $\mathbf{InfL}^{\mathrm{op}}$, or equivalently a functor from $\mathbf{Set}^{\mathrm{op}}$ to **InfL**, which also establishes the equivalence. This in particular means that every topological category is a *bifibration*. The following result is actually a corollary of the general result of what's called *Grothendieck construction*:

Theorem 2.4: The data of a fibre-small topological category $(\mathcal{A}, |-|)$ is the same as the data of a functor $\mathcal{A}_- : \mathbf{Set} \to \mathbf{SupL}$.

Proof We've already seen that any topological category $(\mathcal{A}, |{-}|)$ induces a functor $\mathcal{A}_{\_} : \mathbf{Set} \to \mathbf{SupL}$. On the other hand, given any functor $F : \mathbf{Set} \to \mathbf{SupL}$, we construct a category $\int F$ as follows: Its objects are pairs (X, A) with A being an element in $F(X)$; a morphism $f : (X, A) \to (Y, B)$ is a function $f : X \to Y$, such that $Ff(A) \leq B$. There is an obvious forgetful functor $p : \int F \to \mathbf{Set}$, sending each object (X, A) to the underlying set X. We claim that $(\int F, p)$ is a topological category. To this end, according to Lemma 2.11, we only need to verify that for arbitrary small family of functions $\{f_i : p(X_i, A_i) \to X\}_{i\in I}$, the final lift exists. We show that the final lift is given by $\bigvee_{i\in I} Ff_i(A_i)$ over X. First of all, since we know that for any $i \in I$,

$$Ff_i(A_i) \leq \bigvee_{i\in I} Ff_i(A_i),$$

it follows that each $f_i : p(X_i, A_i) \to p(X, \bigvee_{i\in I} Ff_i(A_i))$ is indeed an $\int F$-morphism. On the other hand, given any $g : p(X, \bigvee_{i\in I} Ff_i(A_i)) \to p(Y, B)$, it is an $\int F$-morphism iff the following holds,

$$Fg \bigvee_{i\in I} Ff_i(A_i) = \bigvee_{i\in I} Fg \circ Ff_i(A_i) = \bigvee_{i\in I} F(g \circ f_i)(A_i) \leq B.$$

The first equality holds because, by definition, Fg is a morphisms in $\mathbf{SupL}$, hence preserves arbitrary joins; the second equality simply follows from the functoriality of F. By the universal property of joins, this in particular means that $g : p(X, \bigvee_{i\in I} Ff_i(A_i) \to p(Y, B)$ is an $\int F$-morphism iff each $g \circ f_i : p(X_i, A_i) \to (Y, B)$ is an $\int F$-morphism, which exactly says that $\bigvee_{i\in I} Ff_i(A_i)$ is the final lift of the structured sink. Hence, $(\int F, p)$ is indeed a topological category. Finally, it is easy to see that the above two processes are mutually inverse to each other. ∎

Thus in later texts, we will not distinguish a topological category $(\mathcal{A}, |{-}|)$, with its associated functor $\mathcal{A}_{(-)} : \mathbf{Set} \to \mathbf{SupL}$, and we will freely change of perspective according to the context. Since we have recorded the fibre structure of all concrete examples of topological categories we have in mind, here we further record how initial and final lifts of single structured sources and sinks could be constructed in them. We will not provide detailed proof of them actually being initial or final lifts of single structured sources and sinks; we leave that for the readers to check, believing that all these constructions are more or less evident:

Example 2.4 (Fibre Connection in **Kr**, **Pre** and **Eqv**): We first look at the most general category **Kr** of arbitrary Kripke frames. Given any function $f : X \to Y$, for any relation R on X, it is easy to see that the final lift of R along f is given by the following relation: For any $y, y' \in Y$, we have $y f_! R y'$ iff there exists x, x' in X, such that $fx = y$ and $fx' = y'$, and that xRx'. In other words, for any $y \in Y$ we have

$$f_! R[y] = f \bigcup_{x \in f^{-1}(y)} R[x].$$

For the initial lift, given any relation Q on Y, it is easy to see for any $x, x' \in X$, we have

$$x f^* Q x' \Leftrightarrow fxQfx'.$$

It is easy to see that, for any preorder or equivalence relation Q on Y, the initial lift f^*Q would again be a preorder or an equivalence relation. Hence, f^* in **Pre** and **Eqv** are described the same as in **Kr**. However, even if R is a preorder or an equivalence relation on X, its final lift along f on Y will not necessarily be a preorder or an equivalence relation: If y does not lie in the image of f, then by definition $f_! R[y]$ in **Kr** would be empty, contradicting to the fact that every preorder and equivalence relation will be at least reflexive. But we can remedy this by taking the reflexive and transitive closure: It is easy to see that the final lift of a preorder R on X in **Pre** is given as follows,

$$f_! R[y] = \left(f \bigcup_{x \in f^{-1}(y)} R[x] \right)^*,$$

where $(-)^*$ denotes the reflexive and transitive closure of a relation. The description of final lifts in **Eqv** again coincide with that of **Pre**, because the reflexive and transitive closure of a symmetric relation is automatically an equivalence relation. ◀

Example 2.5 (Fibre Connections in **Top**): The description of initial lifts of single structured sources is already very well-known in the context of topological spaces. For any function $f : X \to Y$, and for any topology γ on Y, its initial lift along f is given by the following topology on X,

$$f^*\gamma = \{ U \subseteq X \mid U = f^{-1}(V) \,\&\, V \in \gamma \}.$$

The topology $f^*\gamma$ is usually called the *weakly induced topology on X by f*, and this has lots of applications in general topology and in analysis. The final lifts of single structured sinks are less known, but equally natural: For any topology τ on X, its final lift along f

is given as follows,

$$f_!\tau = \{ V \subseteq Y \mid f^{-1}(V) \in \tau \}.$$

In both cases it is easy to verify that $f^*\gamma$ and $f_!\tau$ are well-defined topologies, and they indeed give the initial and final lifts along f, respectively. ◀

Example 2.6 (Fibre Connections in **Mon** and **Nb**): Finally, we look at the initial and final lifts in the category of neighbourhood frames **Nb**, and the subcategory of monotone neighbourhood frames **Mon**. Given any function $f : X \to Y$, and any neighbourhood relation E on X, the final lift of E along f is given as follows: For any $y \in Y$ and $T \subseteq Y$,

$$y f_! E T \Leftrightarrow \forall x \in f^{-1}(y)[x E f^{-1}(T)].$$

It is easy to see that if E is monotone, then so does $f_!E$. Hence, the final lifts of single structured sinks in **Mon** coincide with that in **Nb**.

For initial lifts, suppose we have a neighbourhood relation F on Y. Now for any $x \in X$ and $S \subseteq X$, we have

$$x f^* F S \Leftrightarrow \exists T \subseteq Y[S = f^{-1}(T) \,\&\, f x F T].$$

However, f^*F may not be monotone, even if F is. In **Mon**, we need to monotonise the initial lifts as follows,

$$x f^* F S \Leftrightarrow \exists T \subseteq Y[f^{-1}(T) \subseteq S \,\&\, f x F T].$$

It is easy to verify that this gives the initial lift of monotone neighbourhood frames. ◀

Using Proposition 2.4, we can also easily show other examples of topological categories that are useful in the discourse of modal logic:

Example 2.7 (Topological Category of Evaluation): Let $\mathcal{V} : \mathbf{Set} \to \mathbf{SupL}$ be the functor sending each set X to the complete lattice $\wp(X)^{\mathsf{P}}$, the set of all functions from the set of propositional variables P to $\wp(X)$, with point-wise order. For any function $f : X \to Y$, $\mathcal{V}$ sends it to the function $\exists_f^{\mathsf{P}}$, where $\exists_f : \wp(X) \to \wp(Y)$ sends each subset of X to its image in Y (cf. Section 3.1.1). In other words, an element in $\mathcal{V}_X$ is an evaluation function of the propositional variables in P on the set X. Now for any $p \in \mathsf{P}$

and any family of evaluation functions $\{V_i\}_{i\in I}$ on X, we have

$$\exists^{\mathsf{P}}_f(\bigvee_{i\in I} V_i)(p) = \exists_f \bigcup_{i\in I} V_i(p) = \bigcup_{i\in I} \exists_f V_i(p) = \bigvee_{i\in I} \exists^{\mathsf{P}}_f(V_i)(p).$$

The above uses the fact that joins are computed point-wise in $\wp(X)^{\mathsf{P}}$ for any set X, and $\exists_f$ preserves unions. We can see this more directly by noticing that $\exists^{\mathsf{P}}_f$ has a right adjoint $(f^{-1})^{\mathsf{P}}$, which simply follows from the adjunction $\exists_f \dashv f^{-1}$. Again, see Section 3.1.1 for more details of such adjunctions.

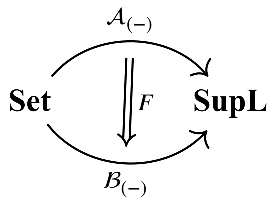
Hence, $\mathcal{V}$ is a topological category. Strictly speaking, the usual types of models of the basic modal language $\mathcal{L}$ is an element in the product of two topological categories $\mathcal{A} \times \mathcal{V}$ with $\mathcal{A}$ being another topological category, e.g. **Kr**, **Top**, etc., such that the part on $\mathcal{V}$ provides the evaluation for propositional variables. Hence, we also call $\mathcal{V}$ the *evaluation topological category*. We will come back to this point in Chapter 3, and introduce some further examples of topological categories there as well. ◀

In fact, the connection between topological categories and functors from **Set** to **SupL** is also functorial. What this means to us is that, there is a bijection between concrete functors between two topological categories which preserves initial or final sources and natural transformations of their associated functors:

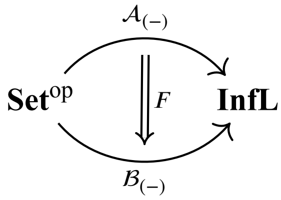
Proposition 2.2: Concrete functors between two topological categories

$$F : (\mathcal{A}, |-|_{\mathcal{A}}) \to (\mathcal{B}, |-|_{\mathcal{B}}),$$

which preserves all final sinks are in one-to-one correspondence with natural transformations of the following type,



Similarly, concrete functors between $\mathcal{A}$ and $\mathcal{B}$ which preserves all initial sources are in one-to-one correspondence with natural transformations of the following type,



Proof We only show the case for concrete functors preserving final sinks; the other case is completely similar. First of all, any concrete functor F induces a family of maps

$$F_X : \mathcal{A}_X \to \mathcal{B}_X,$$

since it preserves the underlying set of each objects in $\mathcal{A}$. This means that, to check it is a natural transformation of some desired type, we only need to show that maps of the form F_X belongs to the codomain category **SupL** or **InfL**, and that naturality holds.

One direction is easy: If F preserves final sinks, then it commutes with final lifts, which means the following diagramme commutes for any $f : X \to Y$,

$$\begin{array}{ccc} \mathcal{A}_X & \xrightarrow{F_X} & \mathcal{B}_X \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ \mathcal{A}_Y & \xrightarrow[F_Y]{} & \mathcal{B}_Y \end{array}$$

Hence, we are left to show that F_X are indeed suplattice morphisms, viz. functions preserving arbitrary joins, for any set X. This also follows from the fact that F preserves final sinks. Given any family $\{A_i\}_{i\in I}$ of objects in $\mathcal{A}_X$, notice that by definition, $\bigvee_{i\in I} A_i$ is the final lift of the structured sink

$$\{1_X : |A_i| \to X\}_{i\in I}.$$

F preserving this final sink amounts to saying that

$$F_X\left(\bigvee_{i\in I} A_i\right) = \bigvee_{i\in I} F_X A_i.$$

Hence, F_X indeed preserves arbitrary joins. This shows that F acts as a natural transformation of the desired type.

On the other hand, given any such natural transformation F, we first show that it indeed induces a concrete functor. Suppose we have a function $f : X \to Y$, such that when considered as a function $f : |A| \to |A'|$ is an $\mathcal{A}$-morphism. To this end, we only need to show that $f : |F_X A| \to |F_Y A'|$ is also a $\mathcal{B}$-morphism. Notice that, $f : |A| \to |A'|$ is an $\mathcal{A}$-morphism iff the following holds,

$$f_! A \le A',$$

which implies that

$$F_Y f_! A = f_! F_X A \le F_Y A'.$$

This uses the fact that F is a natural transformation, hence commutes with final lifts. This gives us the desired results. We also need to show that the induced functor F preserves final sinks, which is easy because the final lift of any structured sink $\{f_i : |A_i| \to X\}_{i\in I}$ is given by $\bigvee_{i\in I}(f_i)_! A_i$. By definition,

$$F_X \bigvee_{i\in I}(f_i)_! A_i = \bigvee_{i\in I} F_X (f_i)_! A_i = \bigvee_{i\in I}(f_i)_! F_{|A_i|} A_i.$$

This uses the fact that F_X is a morphism in **SupL**, hence preserves arbitrary join, and the fact that it commutes with final lifts. Hence, it follows that F preserves final sinks. ∎

Henceforth, if we know a concrete functor F preserves initial sources or final sinks, we will not distinguish between it with its associated natural transformation of the appropriate type. As we will see later, the two different types of naturality involved in the above definition, in the context of modal logic, actually signifies the two types of modal operators that are *dual* to each other. We will come back to this point in Chapter 3.

We end this chapter by applying the above results to characterise when a full subcategory $(\mathcal{B}, |-|)$ of a topological category $(\mathcal{A}, |-|)$ is itself topological. Notice that, if $\mathcal{B}$ is a full subcategory of $\mathcal{A}$, then for any function $f : |B| \to |B'|$, with B, B' in $\mathcal{B}$, it is a $\mathcal{B}$-morphism iff it is an $\mathcal{A}$-morphism, iff one, and hence both, of the following holds,

$$B \le f^* B', \quad f_! B \le B'.$$

Here, $f^*, f_!$ are computed as in $\mathcal{A}$. We first consider the case where the full inclusion further preserves initial sources or final sinks:

Proposition 2.3: Let $\mathcal{B}$ be a concrete full subcategory of a topological category $\mathcal{A}$, then the following two conditions are equivalent:

1. $\mathcal{B}$ is *initially closed*, i.e. for any structured source $\{f_i : X \to |B_i|\}_{i\in I}$ with each B_i in $\mathcal{B}$, the unique initial lift A in $\mathcal{A}$ is also in $\mathcal{B}$;
2. $\mathcal{B}$ is a concrete reflexive category of $\mathcal{A}$.①

Both of the conditions above implies that $\mathcal{B}$ is a topological category. We have a completely dual proposition, which state the equivalence of the following two conditions:

1. $\mathcal{B}$ is *finally closed* in $\mathcal{A}$;
2. $\mathcal{B}$ is a concrete coreflexive subcategory of $\mathcal{A}$.

① The same as our general terminology explained in Remark 2.1 on the use of the world "concrete", a concrete reflexive subcategory $\mathcal{B}$ of $\mathcal{A}$ means that the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ has a concrete right adjoint.

Proof As usually, we only prove the initially closed case, and the other case follows by duality. (1) ⇒ (2): Suppose $\mathcal{B}$ is initially closed in $\mathcal{A}$. By definition of topological categories, this in particular means that $\mathcal{B}$ is also topological, and that the concrete functor $\mathcal{B} \hookrightarrow \mathcal{A}$ preserves initial sources. By Proposition 2.2, it follows that for any set X, the embedding $\mathcal{B}_X \hookrightarrow \mathcal{A}_X$ preserves arbitrary meets, and that for any function $f : X \to Y$, the following diagramme commutes,

$$\begin{array}{ccc} \mathcal{B}_X & \hookrightarrow & \mathcal{A}_X \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathcal{B}_Y & \hookrightarrow & \mathcal{A}_Y \end{array}$$

Since $\mathcal{B}_X \hookrightarrow \mathcal{A}_X$ preserves meets, it has a left adjoint $(-)^\sharp : \mathcal{A}_X \to \mathcal{B}_X$ given by

$$A^\sharp = \bigwedge \{ B \in \mathcal{B}_X \mid B \geq A \},$$

which must be reflexive. Hence, we have fibre-wise left adjoints of the inclusion, and what it remains is to show the functoriality of $(-)^\sharp$. To this end, we need to show that for any function f from X to Y that $f : |A| \to |B|$ is an $\mathcal{A}$-morphism, we would also have a $\mathcal{B}$-morphism $f : \left|A^\sharp\right| \to \left|B^\sharp\right|$. This is immediate from the naturality diagramme above,

$$A \leq f^* B \Rightarrow A^\sharp \leq (f^* B)^\sharp \leq f^* B^\sharp.$$

We have this because, due to the above naturality diagramme, f^* preserves $\mathcal{B}$-objects, hence $f^* B \leq f^* B^\sharp$ implies that $(f^* B)^\sharp \leq f^* B^\sharp$, by the above definition of $(-)^\sharp$. This implies the functoriality of $(-)^\sharp$, and identifies $\mathcal{B}$ as a concrete reflexive subcategory of $\mathcal{A}$.

(2) ⇒ (1): Suppose $\mathcal{B}$ is a concrete reflexive subcategory of $\mathcal{A}$, and we similarly denote the concrete left adjoint as

$$(-)^\sharp : \mathcal{A} \to \mathcal{B}.$$

Given any structure source $\{ f_i : X \to \left|B_i\right| \}_{i \in I}$ in $\mathcal{B}$, we first let A be its initial lift in $\mathcal{A}$. We show that $A^\sharp$ is the initial lift in $\mathcal{B}$. First, for any $i \in I$ the function $f_i : \left|A^\sharp\right| \to \left|B_i\right|$ must be a $\mathcal{B}$-morphism; this is due to the fact that $(-)^\sharp$ is left adjoint to the inclusion and by definition $f_i : |A| \to \left|B_i\right|$ is an $\mathcal{A}$-morphism. On the other hand, given any function $g : |B| \to \left|A^\sharp\right|$ with $B \in \mathcal{B}$, suppose $f_i \circ g : |B| \to \left|B_i\right|$ are all $\mathcal{B}$-morphisms.

Then by definition $g : |B| \to |A|$ is an $\mathcal{A}$-morphism, hence $g : |B| \to \left|A^\sharp\right|$ is also an $\mathcal{A}$-morphism. Since any reflexive subcategory is full, this is also a $\mathcal{B}$-morphism. This completes the proof that $\mathcal{A}^\sharp$ is the initial lift of this structured source, and thus $\mathcal{B}$ is initially closed. ∎

We have many examples of concrete (co)reflexive subcategories of a topological category. For instance, we will show later that **Pre** and **Eqv** are both reflexive subcategories of **Kr**, hence they are both initially closed with respect to **Kr**. Other examples include the embedding of **Mon** into **Nb**. Using the above result, we can characterise the exact condition for a full subcategory itself to be topological:

Theorem 2.5: Suppose $(\mathcal{A}, |-|)$ is a topological category, and $\mathcal{B}$ is a full subcategory of $\mathcal{A}$. The following conditions are equivalent:

1. $(\mathcal{B}, |-|)$ is also a topological category;
2. The inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ has a concrete retract, i.e. a concrete functor $R : \mathcal{A} \to \mathcal{B}$ which fixes objects in $\mathcal{B}$.

Proof (2) ⇒ (1): Suppose we have a concrete retract $R : \mathcal{A} \to \mathcal{B}$. This in particular shows that for any set X, we have a fibre-wise retract

$$R : \mathcal{A}_X \to \mathcal{B}_X.$$

Functoriality means that for any $f : X \to Y$ and any $A \in \mathcal{A}_X, A' \in \mathcal{A}_Y$,

$$A \leq f^* A' \Rightarrow RA \leq f^* RA',$$

or equivalently,

$$f_! A \leq A' \Rightarrow f_! RA \leq RA'.$$

Now for any structured source $\{f_i : X \to |B_i|\}_{i \in I}$ with each B_i in $\mathcal{B}$, we claim that $R \bigwedge_{i \in I} f_i^* B_i$ is the initial lift of it in $\mathcal{B}$. First of all, we know that

$$\bigwedge_{i \in I} f_i^* B_i \leq f_i^* B_i \Rightarrow R \bigwedge_{i \in I} f_i^* B_i \leq f_i^* R B_i = f_i^* B_i,$$

it follows that for each i, $f_i : \left|R \bigwedge_{i \in I} f_i^* B_i\right| \to |B_i|$ is indeed a $\mathcal{B}$-morphism. Now for any $g : |B| \to \left|R \bigwedge_{i \in I} f_i^* B_i\right|$ with B in $\mathcal{B}$, suppose we know that $f_i \circ g : |B| \to |B_i|$ are $\mathcal{B}$-morphisms for all i, i.e. for any $i \in I$ we have

$$B \leq g^* f_i^* B_i.$$

By the fact that g^* is a right adjoint, hence preserves arbitrary meets, it follows that

$$B \leq \bigwedge_{i \in I} g^* f_i^* B_i = g^* \bigwedge_{i \in I} f_i^* B_i \Rightarrow RB = B \leq g^* R \bigwedge_{i \in I} f_i^* B_i.$$

This shows that $g : |B| \to \left|R \bigwedge_{i \in I} f_i^* B_i\right|$ is also a $\mathcal{B}$-morphism, hence $R \bigwedge_{i \in I} f_i^* B_i$ is indeed the initial lift of this structured source, which means $\mathcal{B}$ is a topological category.

(1) $\Rightarrow$ (2): Suppose $\mathcal{B}$ is a full subcategory of $\mathcal{A}$, such that $\mathcal{B}$ is also topological. In particular, for any set map f, we use $f_{\mathcal{B}}^*$ to denote the pullback map induced by f, according to the topological structure in $\mathcal{B}$. We simply use f^* to denote the ordinary pullback map according to the $\mathcal{A}$-topological structure. Since $\mathcal{B}$ is a full subcategory of $\mathcal{A}$, we must have $f^* \leq f_{\mathcal{B}}^*$, for any function f. For any A in $\mathcal{A}_X$, we construct RA as the initial lift of the following structured source *in* $\mathcal{B}$,

$$\{\, 1_X : X \to |B| \mid B \in \mathcal{B}_X \,\&\, A \leq B \,\}.$$

First of all, if A is already an object in $\mathcal{B}$, then obviously RA is A itself. Hence, we are left to show the functoriality of R. Now suppose we have a function $f : X \to Y$ and $A \in \mathcal{A}_X$, $A' \in \mathcal{A}_Y$ such that $f : |A| \to |A'|$ is an $\mathcal{A}$-morphism, viz. $A \leq f^* A'$, then for any $A' \leq B$ in $\mathcal{A}_Y$, we have

$$A \leq f^* A' \leq f^* B \leq f_{\mathcal{B}}^* B.$$

By definition, it follows that $RA \leq f_{\mathcal{B}}^* B$, which means that $f : |RA| \to |B|$ is a $\mathcal{B}$-morphism, for any $A' \leq B$. Since RA' is the initial lift of such structured source, it follows that $f : |RA| \to |RA'|$ must also be a $\mathcal{B}$-morphism, hence R is indeed functorial. ∎

For those readers who want to know more details about topological categories, and its relation with the theory of (bi)fibrations, see relevant sections in[38] and[44].

Chapter 3　Topological Structures Underlying the Landscape

In Chapter 2, we have introduced the mathematical language and topological categories. In this chapter, such a language will be used as the common setting to describe all the main categories of models appearing in the information landscape, and how they relate to different systems of epistemic modal logics, so as to answer Problem 1.1 we raise in Chapter 1.

The approach we have adopted in this chapter is that, we will always first provide descriptions of how different aspects of topological categories could be used as general tools to model different features of modal logic, on a purely abstract level. Section 3.1 will discuss how semantics of basic modal logic could be treated in our framework of topological categories, with the notion of a semantic functor. In particular, we will prove several duality results between the usual categories of semantic models of modal logic we mentioned earlier, and certain categories of algebraic nature. Section 3.2 will look at multi-agent case, and discuss the comparison and dependence between pair of agents represented by modalities. Section 3.3 will put still richer structures in the language, by allowing us to combining a group of agents using the structure of the fibres in a topological category. Finally, Section 3.4 will provide an alternative formulation of all previous structures, by identifying agents as empirical variables in a topological category.

We then proceed, in each subtopic, to further discuss how the general description specialise to the more concrete examples we have in mind. We believe, in this way, topological categories as a conceptualisation, or a unification, of many systems of modal logic in a very wide context, should be self-evident to the readers after completing this chapter.

3.1　Modalities Induced by Topological Structures

We start with the very basics of modal logic. Let us first briefly recall the syntax and semantics of modal logic in its broadest terms. A standard reference for modal logic is[45]. For simplicity, we only concern modalities of arity 1.

Let Σ be a signature of modalities, and let P be a countable set of propositional variables. The modal language $\mathcal{L}_\Sigma$ over the signature Σ and the variable set P is the set of modal formulas recursively defined as follows,

$$\varphi := p \in \mathsf{P} \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Box_a \varphi,$$

where a ranges in Σ. When Σ is a singleton, we will usually omit the subscript. As usual, we treat other logical operators as defined notions:

$$\varphi \vee \psi :\equiv \neg(\neg\varphi \wedge \neg\psi), \quad \varphi \to \psi :\equiv \neg\varphi \vee \psi, \quad \varphi \leftrightarrow \psi :\equiv (\varphi \to \psi) \wedge (\psi \to \varphi).$$

In particular, we have the dual modality $\Diamond_a$ for any $a \in \Sigma$ defined as follows,

$$\Diamond_a \varphi :\equiv \neg\Box_a \neg\varphi.$$

One of the most general types of semantics we can provide for modal logic is one of an algebraic flavour. Here, generally, we define a semantic model $\mathfrak{M}$ for the modal language $\mathcal{L}_\Sigma$ consists of a base set X, equipped with the following additional structures:

- $\mathfrak{M}$ specifies an interpretation function V, which takes each propositional variable p to a subset $V(p) \subseteq X$;
- For each $a \in \Sigma$, $\mathfrak{M}$ specifies an operation $m_a : \wp(X) \to \wp(X)$.①

Given a semantic model $\mathfrak{M}$ with base set X, each formula φ in $\mathcal{L}_\Sigma$ has an *interpretation* $[\![\varphi]\!]_{\mathfrak{M}}$ in $\mathfrak{M}$, which is a subset of X. The interpretation of a formula is inductively defined as follows:

- $[\![p]\!]_{\mathfrak{M}}$ is simply the interpretation $V(p)$;
- $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$;
- $[\![\neg\varphi]\!]_{\mathfrak{M}} = X/[\![\varphi]\!]_{\mathfrak{M}}$;
- $[\![\Box_a \varphi]\!]_{\mathfrak{M}} = m_a([\![\varphi]\!]_{\mathfrak{M}})$

Given such a model $\mathfrak{M}$, we say $\mathfrak{M}$ *satisfies* φ, or $\mathfrak{M}$ is a model of φ, denoted as $\mathfrak{M} \vDash \varphi$, if the interpretation of φ is the total set X,

$$\mathfrak{M} \vDash \varphi \Leftrightarrow [\![\varphi]\!]_{\mathfrak{M}} = X.$$

There is also a local version of this: For any $x \in X$, we write

$$\mathfrak{M}, x \vDash \varphi \Leftrightarrow x \in [\![\varphi]\!]_{\mathcal{M}}.$$

① In most contexts, these operators are often further required to satisfy some additional conditions. For example, normal modal logic requires them to preserve finite meets; most of the epistemic or doxastic systems extend S4, which means the modal operators involved should also be decreasing and idempotent. However, for a general discussion we do not impose any condition on the operators.

Actually, we even can organise the structure of such algebraic data in a semantics of modal logic itself into a concrete category over **Set**. To see this, we first note that there is a very close relationship between complete atomic Boolean algebras and Boolean homomorphisms on one side, and sets and functions on the other side. Due to a famous theorem of Lindenbaum and Tarski, a complete Boolean algebra B is isomorphic to the power set algebra $\wp(X)$ for some set X, iff it is atomic; see[46]. The set X can simply taken to be the set of atoms of B. Furthermore, if $B \cong \wp(X)$ and $B' \cong \wp(X')$, then every Boolean algebra homomorphism $h : B \to B'$ comes from the inverse image $f^{-1} : \wp(X) \to \wp(X')$ of some set map $f : X' \to X$. Hence, we can identify **CABA**, the category of complete atomic Boolean algebras (henceforth, CABAs) and Boolean algebra homomorphisms between them, as the category consists of objects of the form $\wp(X)$ for some set X, and morphisms $f^{-1} : \wp(X) \to \wp(X')$ for some $f : X' \to X$. This shows that **CABA** is isomorphic to the opposite category of **Set**,

$$\mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}}.$$

Notice here that our definition of the category **CABA** is not exactly the same as the usual category of CABA. If we use $\mathbf{CABA}'$ to denote the category of all CABAs, then our category is in fact the image corresponding to the contravariant power set functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{CABA}'$. However, as we've mentioned, every CABA is isomorphic to one in this image, hence we have a categorical equivalence between the category **CABA** as we've described above, and the usual category $\mathbf{CABA}'$. Adopting such a choice is to actually make the inverse of the power set functor concrete,

$$\mathbf{CABA}^{\mathrm{op}} \to \mathbf{Set}.$$

Based on this category **CABA**, for any modal signature Σ we have an induced category of **CABAO**, the category of CABAs *with operators* (henceforth, CABAOs), where its objects are B in **CABA** equipped with an operator $m : B \to B$. In its full generality, we consider *any* endo-function on a CABA, with no further restriction, i.e. we just require m to be a function from B to B. However, as we've mentioned, in most contexts they will be further required to satisfy some further properties. For instance, in most of the systems of modal logic we are dealing at least with monotone modalities.

Which notion of morphisms in **CABAO** should we consider? At first sight, it is very natural to consider complete Boolean algebra homomorphisms which preserve the operators, i.e. $h : B \to B'$ in **CABA** that further satisfies

$$h \circ m = m' \circ h.$$

We call such a function *algebraic* between CABAOs. However, algebraic functions are in some sense too strong for most of the applications we have in mind. In logical terms, such functions will always induce equivalence of satisfaction between models:

Lemma 3.1: Let m, m' be two operators on $\wp(X), \wp(Y)$, respectively, and let $f : X \to Y$ be any function. Recall from Example 2.7 that, the evaluation category $\mathcal{V}$ is also topological: For any evaluation function V on $\wp(Y)$, there is a pullback evaluation f^*V on $\wp(X)$, such that for any propositional letter $p \in \mathsf{P}$,

$$(f^*V)(p) = f^{-1}(V(p)).$$

Then f^{-1} is an algebraic morphism for m, m' iff for any evaluation function V on $\wp(Y)$, and any formula $\varphi \in \mathcal{L}$, the following holds:

$$f^{-1}[\![\varphi]\!]^{V}_{Y,m'} = [\![\varphi]\!]^{f^*V}_{X,m}.$$

Proof For the only if direction, suppose f^{-1} is algebraic. We show the desired property by induction. For the base step, it follows simply from the definition of pullback of evaluation functions. For conjunction, we simply calculate

$$f^{-1}([\![\varphi \wedge \psi]\!]^{V}_{Y,m'}) = f^{-1}([\![\varphi]\!]^{V}_{Y,m'} \wedge [\![\psi]\!]^{V}_{Y,m'}) = [\![\varphi]\!]^{f^*V}_{X,m} \wedge [\![\psi]\!]^{f^*V}_{X,m} = [\![\varphi \wedge \psi]\!]^{f^*V}_{X,m}.$$

The first and last equality follows by definition, and the second equality holds by induction hypothesis. The case for negation and the modal operator proceeds exactly as above, using the fact that f^{-1} preserves negation and the operators.

For the if direction, for any $S \subseteq Y$, we take the constant evaluation function V_S on Y, such that $V_S(p) = S$, for any $p \in \mathsf{P}$. Obviously, the pullback evaluation f^*V_S is just $V_{f^{-1}(S)}$, viz. the constant evaluation function on X sending everything to $f^{-1}S$. Now by definition,

$$f^{-1}[\![\Box p]\!]^{V_S}_{Y,m'} = [\![\Box p]\!]^{f^*V_S}_{X,m} = [\![\Box p]\!]^{V_{f^{-1}S}}_{X,m}.$$

The left hand side by definition equals to $f^{-1}(m'S)$, and the right hand side by definition equals to $m(f^{-1}S)$. Hence, f^{-1} preserves the operators at S. This holds for all such evaluation functions V_S means exactly that f^{-1} is algebraic. ∎

However, it is well-known that, the morphisms in most of the categories listed in

Example 2.1, hence in most concrete categories that we care about, do not induce equivalence of truth between models. Hence, we need to consider a weaker notion of morphisms accordingly on this algebraic level. It turns out that the following weakening is appropriate:

Definition 3.1: Let $h : B \to B'$ be a Boolean algebra homomorphisms between two CABAs, and let m, m' be two operators on them, respectively. h is said to be *continuous* for m, m', for any $x \in B$, we have

$$h \circ m \leq m' \circ h.$$

The order here is just the point-wise order of monotone maps.

Logically speaking, since the definition of continuous morphisms between CABAOs relaxes the preservation of operators to an inequality, we no longer have similar results for a certain equivalence of satisfaction relation between the two models connected by a continuous morphisms. However, we can still say something along these lines. In particular, continuous morphisms always *reflect* the truth for the positive fragment of the modal language, which is easy to see. More importantly, as we will see later, the above definition is the correct algebraic counterpart, simultaneously for monotone functions between relations, continuous functions between topological spaces, and morphisms between evidence frames.

Hence now, we have a well-defined category **CABAO**, with objects being CABAOs and morphisms being continuous Boolean algebra homomorphisms between them. Obviously, $\mathbf{CABAO}^{\mathrm{op}}$ is a construct, with the following evident forgetful functor,

$$\mathbf{CABAO}^{\mathrm{op}} \to \mathbf{CABA}^{\mathrm{op}} \cong \mathbf{Set}.$$

For any set X, the fibre $\mathbf{CABAO}^{\mathrm{op}}_X$ is the set of endo-functions on $\wp(X)$, which is obviously a complete lattice with the point-wise order. This also suggests that $\mathbf{CABAO}^{\mathrm{op}}$ is fibre-small. We further show that $\mathbf{CABAO}^{\mathrm{op}}$ is also a topological category:

Proposition 3.1: Let the above defined functor be $p : \mathbf{CABAO}^{\mathrm{op}} \to \mathbf{Set}$. It makes $\mathbf{CABAO}^{\mathrm{op}}$ into a topological category over **Set**.

Proof We've already seen that $\mathbf{CABAO}^{\mathrm{op}}_X$ is a complete lattice for any set X. Using Theorem 2.4, we need to show that the fibres and final lifts could be made into a functor $\mathbf{Set} \to \mathbf{SupL}$, which send each set X to $\mathbf{CABAO}^{\mathrm{op}}_X$. Explicitly, we only need to show that final lifts of single structured sink exists in $\mathbf{CABAO}^{\mathrm{op}}$, the final lifts preserves arbitrary joins in the fibre of $\mathbf{CABAO}^{\mathrm{op}}$, and that it is functorial.

Suppose we have a function $f : X \to Y$. Let $(\wp(X), m)$ be an object in $\mathbf{CABAO}^{\mathrm{op}}_X$, we define its final lift along f to be the following operator on $\wp(Y)$,

$$f_! m := \forall_f \circ m \circ f^{-1},$$

where $\forall_f$ is the right adjoint of f^{-1} (again, cf. Section 3.1.1). First of all, it is easy to notice that

$$f^{-1} \circ \forall_f \circ m \circ f^{-1} \leq m \circ f^{-1},$$

simply because of the adjunction $f^{-1} \dashv \forall_f$. This shows that $f^{-1} : p(\wp(Y), f_! m) \to p(\wp(X), m)$ is indeed a continuous map, hence it is a $\mathbf{CABAO}^{\mathrm{op}}$-morphism from $(\wp(X), m)$ to $(\wp(Y), f_! m)$. Furthermore, for any function $g : Y \to Z$ and any operator n on $\wp(Z)$, by the adjunction again we have

$$f^{-1} \circ g^{-1} \circ n \leq m \circ f^{-1} \circ g^{-1} \Leftrightarrow g^{-1} n \leq \forall_f \circ m \circ f^{-1} \circ g^{-1}.$$

This shows that g^{-1} is continuous for $n, f_! m$ iff $(gf)^{-1}$ is continuous for m, n. Hence, $f_! m$ as defined above is indeed the final lift.

We then show that $f_!$ preserves all joins in $\mathbf{CABAO}^{\mathrm{op}}_X$, which is equivalent to say that it preserves all meets in $\mathbf{CABAO}_X$. To this end, we observe that meets in $\mathbf{CABAO}_X$ is calculated as point-wise intersections, and that $\forall_f$ is a right adjoint, hence preserves arbitrary intersections. Hence, for any family $\{m_i\}_{i \in I}$ in $\mathbf{CABAO}_X$, we have

$$f_! \left(\bigcap_{i \in I} m_i \right) = \forall_f \circ \bigcap_{i \in I} m_i \circ f^{-1} = \bigcap_{i \in I} \forall_f \circ m_i \circ f^{-1} = \bigcap_{i \in I} f_! m_i.$$

Hence, $f_!$ indeed preserves arbitrary meets, and it can be viewed as a morphism $f_! : \mathbf{CABAO}^{\mathrm{op}}_X \to \mathbf{CABAO}^{\mathrm{op}}_Y$ in $\mathbf{SupL}$.

Finally, the construction is obviously functorial, since we have

$$g_! f_! m = g_!(\forall_f \circ m \circ f^{-1}) = \forall_g \circ \forall_f \circ m \circ f^{-1} \circ g^{-1} = \forall_{gf} \circ m \circ (gf)^{-1} = (gf)_! m.$$

This uses the functoriality of $(-)^{-1}$, and the fact that $\forall_{(-)}$ assigns right adjoints, hence is also functorial. This completes the proof that $\mathbf{CABAO}^{\mathrm{op}}$ is a topological category. ∎

From the above proof, we already know that for any function $f : X \to Y$, the final lift along f in $\mathbf{CABAO}^{\mathrm{op}}$ is given by

$$m \mapsto \forall_f \circ m \circ f^{-1}.$$

We know that initial lifts and final lifts form the following adjunction,

$$f_! : \mathbf{CABAO}_X^{\mathrm{op}} \rightleftarrows \mathbf{CABAO}_Y^{\mathrm{op}} : f^*.$$

We would also like an explicit description of f^*, but since we work with arbitrary operators on CABAs, not even monotone ones, there is no single simple formula for the initial lift f^*. The best we can get is the general formula for calculating adjoins for complete lattices: For any operator n on $\wp(Y)$, we have

$$f^* n = \bigcap \{ m \mid n \leq f_! m \}.$$

Notice that, the reason we have this is that, when considering $f_!$ and f^* as morphisms between $\mathbf{CABAO}_X$ and $\mathbf{CABAO}_Y$, the left and right adjoints must switch, and we actually have $f^* \dashv f_!$. This suggest, in some sense, that considering final lifts in $\mathbf{CABAO}^{\mathrm{op}}$ is more natural than considering initial lifts. This is also why we focus on $\Box$, not $\Diamond$, for the basic operators of modal logic.

However, the usual familiar semantics for modal logic are often not of such an algebraic form, but most of them can be transformed into this style, since the algebraic formulation of semantics is clearly what's minimal to obtain an interpretation for modal formulas in this general context. In other words, we may define the topological categories that can be used to provide models for modal algebras as those ones equipped with a certain transformation to **CABAO**. This works even more generally for ordinary construct:①

Definition 3.2: For a construct $(\mathcal{A}, |-|)$, a *semantic functor* is a concrete functor of the following type,

$$(-)^+ : \mathcal{A} \to \mathbf{CABAO}^{\mathrm{op}}.$$

The functor $(-)^+$ being concrete means that, for any A in $\mathcal{A}$, it is sent to a CABAO of the form $(\wp(|A|), m_A)$, where m_A is an operator on $\wp(|A|)$. In most of the cases, $(-)^+$ will be *injective on objects*, which in particular means that, for any set X, the following

① However, other features of modal logic do requires more structures in topological categories than ordinary constructs. We will discuss them in subsequent sections in this chapter.

induced monotone map between fibres is an embedding,

$$\mathcal{A}_X \to \mathbf{CABAO}_X^{\mathrm{op}}.$$

By Lemma 2.1, such an assignment defines a concrete functor only if for any $f : |A| \to |B|$ which is an $\mathcal{A}$-morphism, the following inverse image function must be continuous,

$$f^{-1} : (\wp(|B|), m_B) \to (\wp(|A|), m_A).$$

When such a functor is present, for any formula φ in the language $\mathcal{L}$, we can already define the interpretation of φ with respect to (A, V), with V being an interpretation function on $|A|$, as follows,

$$[\![\varphi]\!]_A^V := [\![\varphi]\!]_{A^+}^V.$$

In particular, $[\![\varphi]\!]_A^V$ would also be a subset of $|A|$. As we will see soon, all our mentioned examples of semantics for different systems of modal logic arise in such a way.

In some concrete examples of modal systems we encounter in the literature, e.g. in some epistemological or doxastic interpretations of modal logic as the examples we have in mind, their semantic functors will have much nicer properties.

First of all, in different contexts, the operators involved are further confined into some subclass. In extremely well-behaved cases, the semantic functor $(-)^+$ will establish a *full embedding* of $\mathcal{A}$ into $\mathbf{CABAO}^{\mathrm{op}}$. In these cases, we can always restricts the type of operators we are considering into a subclass C, and let $\mathbf{CABAO}_C$ denote the full subcategory of **CABAO**, consisting of those CABAOs with operators in this class. If we identify C with the those operators that lie in the image of the semantic functor $(-)^+$, and when $(-)^+$ is a full embedding, then we would have a *concrete isomorphism* of the following type,

$$(-)^+ : \mathcal{A} \cong \mathbf{CABAO}_C^{\mathrm{op}}.$$

In such a case, $\mathbf{CABAO}_C^{\mathrm{op}}$ will automatically be a topological category as well if $\mathcal{A}$ is, since the above is a concrete isomorphic between the two. This means that there is a concrete *duality* between the category of semantics, which signifies our geometric intuition about modal logic on one hand, and the category of a class of algebraic entities, which represents the abstract mathematical structure of the syntax and inference systems of modal logic on the other hand. This is the most desirable scenario, according to our general philosophy outlined in Section 1.4, and indeed, as we will see, most examples of the modal systems we consider have this nice properties.

In particular, when the above semantic functor is indeed an equivalence, we can immediately conclude that the semantic category $\mathcal{A}$ is not only sound, but also *complete* for the class C.① This simply follows from the Lindenbaum-Tarski algebra construction and the Jónsson-Tarski canonical extension theorem. Details could be found in most modal logic textbooks, e.g. in[45] or in the handbook[35]. However, analysing completeness of various fragments of modal logic is not the main subject of this paper, hence we will only briefly comment on this when such a duality principle is present.

As an example, a commonly seen requirement in the context for epistemology is for the operator to be an *interior operator*, viz. a decreasing, monotone, idempotent function. In modal logic terminology, these operators are said to satisfy the **S4** axioms. We then have a full subcategory $\mathbf{CABAO}_{\mathcal{I}}$ of **CABAO**, consisting of complete atomic Boolean algebras with interior operators. If our proposed category of semantic models $\mathcal{A}$ respects **S4**, or is sound for **S4** logic, then the concrete semantic functor $(-)^+$ must restrict to the following type,

$$(-)^+ : \mathcal{A} \to \mathbf{CABAO}_{\mathcal{I}}^{\mathrm{op}}.$$

Built on the above description, we can also show how duality of modalities could be treated in our general framework. Notice that there is a natural dual operation on the category **CABAO**. We write $\sim$ for the operation of taking the complement of a subset, then for any operator m on $\wp(X)$, we define its dual operator m° as follows,

$$m^\circ := \sim \circ m \circ \sim .$$

Notice that, m is monotone iff m° is. The operation $(-)^\circ$ is an *involution*, because $\sim$ is,

$$(m^\circ)^\circ = \sim \circ \sim \circ m \circ \sim \circ \sim = m.$$

This means that the dual operation consists of a concrete automorphism on the category **CABAO**,

$$(-)^\circ : \mathbf{CABAO} \cong \mathbf{CABAO}.$$

More generally, for any subclass C of operators, we can also define C° to be its dual class, i.e. those operators m whose dual operator m° is in C. Similarly we have $(C^\circ)^\circ = C$, and the above automorphism on **CABAO** restricts to a concrete isomorphism of the following

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① However, this in general is not a necessary condition for completeness.

type,

$$(-)^{\circ} : \mathbf{CABAO}_{C} \cong \mathbf{CABAO}_{C^{\circ}}.$$

Immediately, we know that $\mathbf{CABAO}_{C}^{\mathrm{op}}$ is a topological category, iff $\mathbf{CABAO}_{C^{\circ}}^{\mathrm{op}}$ is, because they are concretely isomorphic. Now for any semantic functor that restricts to the subclass C, by composing with this duality operation, we get another semantic functor of the following type,

$$\mathcal{A} \xrightarrow{(-)^{+}} \mathbf{CABAO}_{C}^{\mathrm{op}} \xrightarrow{(-)^{\circ}} \mathbf{CABAO}_{C^{\circ}}^{\mathrm{op}},$$

which takes an object A in $\mathcal{A}_X$ to its dual operator on $\wp(X)$ given by $(-)^{+}$, for any set X.

As we've mentioned, our specific choice of the notion of morphisms in **CABAO** is to make the above definition fits for most of the examples we have in mind. Later on in this section, we will describe what the semantic functors look like for the examples we have in mind, including **Kr**, **Pre**, **Eqv**, **Top**, **Nb** and **Mon**. In particular, the usual morphisms there in those categories corresponds exactly to continuous morphisms between CABAOs. However, a natural question is then to characterise those morphisms in these categories that are actually mapped into algebraic morphisms between CABAOs. These morphisms then necessarily induce equivalent satisfaction relation between the two models, due to Lemma 3.1. This recovers the usual bounded morphisms or p-morphisms in **Kr**, **Pre** and **Eqv**, continuous open maps in **Top**, and bounded morphisms in **Nb** and **Mon**.

Remark 3.1: Strictly speaking, the category **CABA** of complete atomic Boolean algebras, and the induced category **CABAO** of complete atomic Boolean algebras with operators, are certain compromised categories between concreteness on one hand, and the algebraic generality on the other hand. As we will see later in Section 3.1.3, we actually have a duality between arbitrary neighbourhood frames and complete atomic Boolean algebras with operators, in that we have a concrete equivalence of the following form,

$$\mathbf{Nb} \cong \mathbf{CABAO}^{\mathrm{op}}.$$

Thus, in some sense, both **CABAO** and **Nb** represent the most general form of semantics we would like to consider in this thesis.

However, it is certainly possible to use the same framework of topological categories to incorporate more general type of algebraic semantics, based on genuine Boolean algebras with operators, otherwise known as BAOs. Just as we have explored the duality between **CABA** and **Set**, to set out the investigation of the more general type of algebraic semantics we need to exploit the duality between **BA**, the category of Boolean algebras, and **Stone**, the category of stone spaces. The statement that these two categories are dual to each other is the famous Stone duality; see[47] for a complete treatment.

In other words, we need to consider topological categories not over **Set**, but over **Stone**. Most of the results we stated in Chapter 2 also applies in the more general context of topological categories over an arbitrary complete and cocomplete categories, which is certainly satisfied by **Stone**. Then $\mathbf{BAO}^{\mathrm{op}}$, the opposite category of the category of Boolean algebras with operators and continuous morphisms between them, will also be topological over **Stone**, and it will play a similar role as $\mathbf{CABAO}^{\mathrm{op}}$ has played here. Such an approach also has the potential of finding new semantic categories of modal logic by exploring the duality between certain subclass of **BAO**, and some geometric category of stone spaces with additional structures. However, to remain as concrete as possible and connect to the usual discourse of ordinary semantics of modal logic, we will remain to only consider topological categories over **Set**.

Furthermore, the adoption of the category $\mathbf{CABAO}^{\mathrm{op}}$ also suggests that all the concrete examples of categories of semantics of modal logic described in this work actually support the infinitary fragment of modal logic, where we allow infinite disjunctions and conjunctions, because $\wp(X)$ is always a complete lattice for any set X. This is another difference between the more algebraic approach, using the topological category of **BAO** over **Stone**. However, as we will see as the texts unfold, the majority of examples of semantics of modal logic we encounter in practice are of this kind. Hence, currently we consider such a choice fit to our general motivation for carrying out this project. ◀

With this abstract formulation in hand, we can now finally look at how concrete examples of semantics of modal logic fall into the general description here.

3.1.1 Semantic Functors for **Kr**, **Pre** and **Eqv**

We first look at the normal relational modal logics, and construct the associated semantic functor for the category of Kripke frames,

$$(-)^+ : \mathbf{Kr} \to \mathbf{CABAO}^{\mathrm{op}}.$$

For any relation $R \subseteq X \times Y$ we have two induced maps $\exists_R, \forall_R : \wp(X) \to \wp(Y)$ as follows,

$$\exists_R(S) = \{ y \in Y \mid \exists x \in S[xRy] \},$$
$$\forall_R(S) = \{ y \in Y \mid \forall x[xRy \to x \in S] \}.$$

We write $R^\dagger$ for the dual relation $R^\dagger \subseteq Y \times X$ of R, and use $R[x]$ to denote the R-image of x, viz. the set $\exists_R(\{x\})$. It is then easy to see that

$$\exists_R(S) = \bigcup_{x \in S} R[x], \quad \forall_R(S) = \{ y \mid R^\dagger[y] \subseteq S \}.$$

From this description, the following is immediate:

Lemma 3.2: For any binary relation $R \subseteq X \times Y$, we have an induced adjunction

$$\exists_R \dashv \forall_{R^\dagger}.$$

Proof For any subsets $S \subseteq X$ and $T \subseteq Y$, we have

$$\exists_R(S) \subseteq T \Leftrightarrow \forall x \in S[R[x] \subseteq T] \Leftrightarrow S \subseteq \{ x \mid R[x] \subseteq T \} \Leftrightarrow S \subseteq \forall_{R^\dagger}(T). \qquad \blacksquare$$

If the relation is functional, i.e. we have an actual function $f : X \to Y$ viewed as a relation, then it is easy to see that $\exists_{f^\dagger} = \forall_{f^\dagger} = f^{-1}$. This implies that $\exists_f \dashv f^{-1} \dashv \forall_f$.①

For any object in **Kr**, viz. a set X equipped with a relation R on it, we can associate it with the following operation

$$\forall_{R^\dagger} : \wp(X) \to \wp(X).$$

By definition, for any $x \in X$ and $S \subseteq X$, we have

$$x \in \forall_{R^\dagger}(S) \Leftrightarrow \forall y[xRy \Rightarrow y \in S].$$

This exactly corresponds to the usual interpretation of modal operators in relational frames. We show that this construction extends to a concrete functor.

① This fact is the starting point of a much more general treatment of quantifiers in categorical logic.

Proposition 3.2: The assignment $(X, R) \mapsto (\wp(X), \forall_{R^\dagger})$ gives a well-defined semantic functor

$$(-)^+ : \mathbf{Kr} \to \mathbf{CABAO}^{\mathrm{op}}.$$

Proof Suppose we have two kripke frames (X, R) and (Y, Q), and a monotone function $f : X \to Y$. For any subset $S \subseteq Y$ and any $x \in X$ we have

$$x \in f^{-1}(\forall_{Q^\dagger}(S)) \Leftrightarrow f(x) \in \forall_{Q^\dagger}(S) \Leftrightarrow Q[f(x)] \subseteq S.$$

Since f is monotone, it follows that

$$f(R[x]) \subseteq Q[f(x)] \subseteq S,$$

which implies that $R[x] \subseteq f^{-1}(S)$, and thus $x \in \forall_{R^\dagger}(f^{-1}(S))$. This implies that

$$f^{-1} \circ \forall_{Q^\dagger} \subseteq \forall_{R^\dagger} \circ f^{-1},$$

which means the inverse image f^{-1} is continuous for the two operators $\forall_{R^\dagger}, \forall_{Q^\dagger}$. ∎

The semantic functor recovers the usual relational semantics of modal logic in Kripke frames. The key point is the following clause: For any Kripke frame (X, R), any $x \in X$, any evaluation function V on X, and any $\varphi \in \mathcal{L}$,

$$\begin{aligned} X, R, V, x \vDash \Box\varphi &\Leftrightarrow x \in [\![\Box\varphi]\!]^V_{(X,R)^+} \\ &\Leftrightarrow x \in \forall_{R^\dagger}[\![\varphi]\!]^V_{(X,R)^+} \\ &\Leftrightarrow \forall y. xRy \Rightarrow y \in [\![\varphi]\!]^V_{(X,R)^+}. \end{aligned}$$

With this in hand, we can directly characterise which types of morphisms in **Kr** correspond to algebraic morphisms between CABAOs. Recall that a monotone function $f : (X, R) \to (Y, Q)$ is said to be a *p-morphism*, or *bounded morphism*, if for any $x \in X, u \in Y$, if $f(x)Qu$, then there exists $y \in X$ such that $f(y) = u$ and xRy. The following shows that p-morphisms are exactly the class of morphisms corresponding to algebraic morphisms between CABAOs, along the semantic functor described above:

Proposition 3.3: For any monotone function $f : (X, R) \to (Y, Q)$, it corresponds to an algebraic morphism between $(\wp(X), \forall_{R^\dagger})$ and $(\wp(Y), \forall_{Q^\dagger})$ iff it is a p-morphism.

Proof Notice that $\forall_{R^\dagger}, \forall_{Q^\dagger}$ and f^{-1} are all right adjoints. f^{-1} is an algebraic morphism, or in other words the following diagramme commutes,

$$\begin{array}{ccc} \wp(Y) & \xrightarrow{f^{-1}} & \wp(X) \\ {\scriptstyle \forall_{Q^\dagger}}\downarrow & & \downarrow{\scriptstyle \forall_{R^\dagger}} \\ \wp(Y) & \xrightarrow[f^{-1}]{} & \wp(X) \end{array}$$

iff the diagramme of their left adjoints commute, i.e. the following equality holds,

$$\exists_R \circ \exists_f = \exists_f \circ \exists_Q.$$

Since left adjoints preserves union, we only need to check this for singletons of the form $\{x\}$ for any $x \in X$. To this end, the above equality translates to the fact that

$$f(R[x]) = Q[f(x)],$$

which is exactly saying that if $f(x)Qu$ for some u, then there exists xRy that $f(y) = u$, i.e. f is a p-morphism. ∎

In fact by Lemma 3.2, $\forall_{R^\dagger}$ is a right adjoint, hence it preserves arbitrary meets. This shows that the semantic functor for **Kr** restricts to the following one,

$$(-)^+ : \mathbf{Kr} \to \mathbf{CABAO}_\wedge^{\mathrm{op}},$$

where the category $\mathbf{CABAO}_\wedge$ is the category of CABAOs whose operators preserve arbitrary meets. It turns out, this is further more an equivalence. To see this, we first observe the following result:

Lemma 3.3: Any map $h : \wp(Y) \to \wp(X)$ that preserves arbitrary meets comes from a relation $R \subseteq X \times Y$ that $h = \forall_{R^\dagger}$.

Proof It is well-known that such h has a left adjoint g. We then define a relation R on X, Y as follows: For any $x \in X$ and $y \in Y$, we set

$$xRy \Leftrightarrow y \in g(\{x\}).$$

We then verify that $\forall_{R^\dagger}$ coincide with h: For any subset $S \subseteq Y$ and any $x \in X$,

$$x \in \forall_{R^\dagger} S \Leftrightarrow R[x] \subseteq S \Leftrightarrow g(\{x\}) \subseteq S \Leftrightarrow x \in h(S).$$

The last step uses the fact that g is a left adjoint of h. Hence, this shows that $\forall_{R^\dagger}$ and h coincide. Such relation R is necessarily unique, hence this completes the proof. ∎

Let **Rel** denote the category whose objects being sets, and whose morphisms from X to Y being binary relations on $X \times Y$. Let $\mathbf{CABA}_\wedge$ be the category of CABAs with arbitrary meet preserving maps between them. In particular, **CABA** is a wide subcategory of $\mathbf{CABA}_\wedge$. Lemma 3.3 implies that we have an equivalence between **Rel** and $\mathbf{CABA}_\wedge$:

Corollary 3.1: Let $\forall_\_ : \mathbf{Rel} \to \mathbf{CABA}_\wedge$ be the functor, which takes each set X to $\wp(X)$ and any relation $R \subseteq X \times Y$ to $\forall_R : \wp(X) \to \wp(Y)$. It defines an equivalence of categories.

Proof It is obvious that $\forall_\_$ is surjective on objects, and by Lemma 3.3 it is also fully faithful. Hence, $\forall_\_$ indeed forms an equivalence of categories. ∎

As we will see, the above equivalence $\mathbf{Rel} \cong \mathbf{CABA}_\wedge$ is at the heart of all the dualities stated in this section. First, we have the duality of Kripke frames:

Proposition 3.4: The semantic functor of Kripke frames restricts to a concrete equivalence of categories,

$$(-)^+ : \mathbf{Kr} \cong \mathbf{CABAO}_\wedge^{\mathrm{op}}.$$

Proof Lemma 3.3 shows that the semantic functor is bijective on objects when the codomain is restricted to $\mathbf{CABAO}_\wedge^{\mathrm{op}}$. The rest follows from Lemma 2.1. ∎

This suggests that $\mathbf{CABAO}_\wedge^{\mathrm{op}}$ is also a topological category, since it is concretely isomorphic to **Kr**, whose identity as a topological category has been established in Corollary 2.3. However, there is another way of seeing this fact which provides even more information:

Proposition 3.5: $\mathbf{CABAO}_\wedge^{\mathrm{op}}$ is finally closed as a full subcategory of $\mathbf{CABAO}^{\mathrm{op}}$.

Proof By Lemma 2.13, we only need to show that $\mathbf{CABAO}_\wedge^{\mathrm{op}}$ is closed under forming arbitrary meets and final lifts $f_!$ for any function f in **CABAO**. For any family of meet-preserving operators $\{m_i\}_{i\in I}$ on $\wp(X)$, for any family of subsets $\{S_j\}_{j\in J}$ of X, we obviously have

$$\left(\bigcap_{i\in I} m_i\right)\bigcap_{j\in J} S_j = \bigcap_{i\in I} m_i\left(\bigcap_{j\in J} S_j\right) = \bigcap_{i\in I}\bigcap_{j\in J} m_i(S_j) = \bigcap_{j\in J}(\bigcap_{i\in I} m_i)(S_j).$$

This uses the fact that meets commute with meets, and that each operator m_i preserves meets. Also, for any meet-preserving operator m on $\wp(X)$ and any function $f : X \to Y$, by the proof of Proposition 3.1, its final lift is given by the following formula,

$$f_! m = \forall_f \circ m \circ f^{-1}.$$

Since both f^{-1} and $\forall_f$ are right adjoints, they also preserve arbitrary meets. This shows that $f_! m$ also preserves meets, hence belongs to $\mathbf{CABAO}_{\wedge}^{\mathrm{op}}$. This completes the proof that it is finally closed. ∎

Corollary 3.2: The inclusion $\mathbf{CABAO}_{\wedge}^{\mathrm{op}} \hookrightarrow \mathbf{CABAO}^{\mathrm{op}}$, hence the semantic functor $(-)^{+} : \mathbf{Kr} \to \mathbf{CABAO}^{\mathrm{op}}$, preserves all final sinks.

Proof Immediate from Proposition 3.5 and the duality shown in Proposition 3.4. ∎

The duality results presented in Proposition 3.4, combined with certain embedding theorems, provides an immediate proof for the completeness of Kripke frame semantics for normal modal logic①. Now when we look at the two full subcategories **Pre** and **Eqv** of **Kr**, they inherits the semantic functor from **Kr**, and they corresponds to smaller classes of operators. These results similarly imply completeness results for some corresponding fragments of modal logics, which we will not repeat again.

Proposition 3.6: Let $\mathbf{CABAO}_{\wedge,\mathcal{I}}$ be the full subcategory of $\mathbf{CABAO}_{\wedge}$ whose operators are furthermore decreasing and idempotent; let $\mathbf{CABAO}_{\wedge,\mathcal{I}_c}$ be the category full subcategory of $\mathbf{CABAO}_{\wedge,\mathcal{I}}$ whose operators are also Euclidean. Then the semantic functor restricts to the following two concrete equivalences:

$$(-)^{+} : \mathbf{Pre} \cong \mathbf{CABAO}_{\wedge,\mathcal{I}}^{\mathrm{op}}, \quad (-)^{+} : \mathbf{Eqv} \cong \mathbf{CABAO}_{\wedge,\mathcal{I}_c}^{\mathrm{op}}.$$

Proof Once we've established the duality stated in Proposition 3.4, these further correspondence between properties of relations on the one hand, and conditions of the operators on the other hand, can be thought as a corollary of modal correspondence theory (cf. the original paper[48] or textbooks[8,45]). ∎

Remark 3.2: By Proposition 3.3, it is possible to state yet a further duality between $\mathbf{Kr}_p$, the category of Kripke frames with p-morphisms, and Alg-$\mathbf{CABAO}_{\wedge}^{\mathrm{op}}$, the category of CABAOs with meet-preserving operators, and *algebraic morphisms* between them. However, this should be viewed as a pure corollary of the more fundamental fact stated in Proposition 3.4. Hence, we do not list it here. ◀

① This holds even for a suitable infinitary modal logic allowing infinite conjunction and disjunction, with the necessity modal operator required to commute with all infinite conjunctions.

3.1.2 Semantic Functor on **Top**

Similarly, in this subsection we want to construct the associated semantic functor for the category of topological spaces,

$$(-)^+ : \mathbf{Top} \to \mathbf{CABAO}^{\mathrm{op}}.$$

In some sense, this is easier than the case on Kripke frames, since it is well-known that the data of a topology on a set X is equivalent to the data of an operator on the power set $\wp(X)$ of a particular kind. Explicitly, for any topology τ on a set X, it induces an operator

$$j_\tau : \wp(X) \to \wp(X),$$

sending each subset S to the largest open set contained in S. In particular, j_τ is a *topological interior operator* on $\wp(X)$, i.e. it is decreasing, monotone, idempotent, and preserves finite intersections. It is well-known that topological interior operators on $\wp(X)$ are in bijective to topologies on X. The open sets induced by a topological interior operator j on $\wp(X)$ is the set of fixed-points of j, viz. the set $\{ j(S) \mid S \subseteq X \}$.

In fact, more is true, and the following result in some sense justifies our terminology for continuous maps between two CABAOs:

Proposition 3.7: For any function $f : (X, \tau) \to (Y, \gamma)$, it is a continuous function between two topological spaces iff it is continuous for the two induced CABAOs $(\wp(X), j_\tau)$ and $(\wp(Y), j_\gamma)$, i.e. iff the following holds,

$$f^{-1} \circ j_\gamma \subseteq j_\tau \circ f^{-1}.$$

Proof Suppose f is a continuous function between two topological spaces. We know that $j_\gamma(T) \subseteq T$, thus $f^{-1}(j_\gamma(T)) \subseteq f^{-1}(T)$. Since $f^{-1}(j_\gamma(T))$ is open for τ since f^{-1} preserves opens, it follows that $f^{-1}(j_\gamma(T)) \subseteq j_\tau(f^{-1}(T))$.

On the other hand, suppose f^{-1} is continuous for the two CABAOs $(\wp(X), j_\tau)$ and $(\wp(Y), j_\gamma)$. Then for any open subset T of Y, we have

$$f^{-1}(T) = f^{-1}(j_\gamma(T)) \subseteq j_\tau(f^{-1}(T)).$$

However, since j_τ is decreasing, it follows that $f^{-1}(T) = j_\tau(f^{-1}(T))$, which exactly means $f^{-1}(T)$ is open. Hence, f is continuous for the two topological spaces. ∎

In particular, the above result shows that we have a concrete equivalence of the type

$$(-)^{+} : \mathbf{Top} \cong \mathbf{CABAO}_{l\mathcal{I}}^{\mathrm{op}},$$

where $\mathbf{CABAO}_{l\mathcal{I}}$ is the full subcategory of **CABAO**, which consists of those CABAOs with a topological interior operator. Again, this further implies that $\mathbf{CABAO}_{l\mathcal{I}}^{\mathrm{op}}$ is also a topological category. This provides the semantic functor for the topological category **Top**, and again it is easy to see that this semantic functor recovers the usual topological semantics for modal logic, by interpreting modalities as interior operators.

It is also easy in this case to characterise the type of morphisms in **Top** corresponding to algebraic morphisms between CABAOs under this semantic functor:

Proposition 3.8: A continuous function $f : (X, \tau) \to (Y, \gamma)$ induces an algebraic map between $(\wp(X), j_\tau)$ and $(\wp(Y), j_\gamma)$ iff it is also open, i.e. f takes open sets in X to open sets in Y.

Proof Every topology τ on X induces an adjunction $i \dashv j_\tau$, where i is simply the inclusion of τ into $\wp(X)$. Now similar to proof of Proposition 3.3, f^{-1} being an algebraic morphism between the two CABAO means the following diagramme commute,

$$\begin{array}{ccc} \wp(Y) & \xrightarrow{f^{-1}} & \wp(X) \\ {\scriptstyle j_\gamma}\downarrow & & \downarrow{\scriptstyle j_\tau} \\ \gamma & \xrightarrow[f^{-1}]{} & \tau \end{array}$$

which is again equivalent to saying that their left adjoints commute, i.e. the following holds,

$$\begin{array}{ccc} \wp(Y) & \xleftarrow{f} & \wp(X) \\ \uparrow & & \uparrow \\ \gamma & \xleftarrow[f]{} & \tau \end{array}$$

This is exactly saying that f is also an open map. ∎

In particular, $\mathbf{CABAO}_{l\mathcal{I}}$ contains $\mathbf{CABAO}_{\wedge,\mathcal{I}}$ as a full subcategory, where operators are required not only to preserve finite meets, but arbitrary ones. Such an observation suggests that there is a fully faithful embedding of the category of preorders into the category of topological spaces,

$$\mathbf{Pre} \cong \mathbf{CABAO}_{\wedge,\mathcal{I}}^{\mathrm{op}} \hookrightarrow \mathbf{CABAO}_{l\mathcal{I}}^{\mathrm{op}} \cong \mathbf{Top}.$$

As we will see in Chapter 4, this embedding corresponds to a well-known construction of associating any preorder a topology, called the Alexandroff topology.

3.1.3 Semantic Functors for **Nb** and **Mon**

Lastly, we describe a semantic functor for the category of neighbourhood frames,

$$(-)^+ : \mathbf{Nb} \to \mathbf{CABAO}^{\mathrm{op}}.$$

Recall from Example 2.1 (v). that a neighbourhood frame is a set X equipped with a relation $E \subseteq X \times \wp(X)$; a morphism $f : (X, E) \to (Y, F)$ is a function from X to Y that further satisfies, whenever $fxFT$ for $x \in X, T \subseteq Y$, we also have $xEf^{-1}(T)$.

Now for every neighbourhood frame (X, E) we can associate it with an operator n_E on $\wp(X)$: For any subset $S \subseteq X$, we define

$$n_E(S) = \{ x \mid xES \}.$$

Notice that, in this case, the operator n_E is not always monotone. In fact, it is monotone iff E is monotone as a neighbourhood relation, i.e. (X, E) lies in **Mon**. A similar result happens yet again:

Proposition 3.9: For any function $f : (X, E) \to (Y, F)$, it is a morphisms between the two neighbourhood frames iff it is continuous for the induced operators n_E, n_F.

Proof For any subset $T \subseteq Y$, we have that

$$f^{-1}(n_F(T)) = \{ x \mid fxFT \},$$

and that

$$n_E(f^{-1}(T)) = \{ x \mid xEf^{-1}(T) \}.$$

Now that f^{-1} is continuous for n_E, n_F means that $f^{-1}(n_F(T)) \subseteq n_E(f^{-1}(T))$, which exactly says that $fxFT$ implies $xEf^{-1}(T)$. ∎

Again, this gives us a well-defined semantic functor

$$(-)^+ : \mathbf{Nb} \to \mathbf{CABAO}^{\mathrm{op}},$$

which, in fact, is an *isomorphism* of categories:

Proposition 3.10: The semantic functor $(-)^+ : \mathbf{Nb} \to \mathbf{CABAO}^{\mathrm{op}}$ gives us a concrete isomorphism of the two concrete categories.

Proof To this end, we only need to show that fibre-wise for any set X, $\mathbf{Nb}_X$ is bijective to $\mathbf{CABAO}^{\mathrm{op}}_X$. For any neighbourhood frame (X, E), we already have an associated operator n_E on $\wp(X)$. On the other hand, given any operator n on $\wp(X)$, we define a relation $E_n \subseteq X \times \wp(X)$ as follows: For any $x \in X$ and $S \subseteq X$,

$$xE_nS \Leftrightarrow x \in n(S).$$

Now we can calculate as follows: For any operator m on $\wp(X)$,

$$x \in n_{E_m}(S) \Leftrightarrow xE_mS \Leftrightarrow x \in m(S).$$

This means that $n_{E_m} = m$. Similarly, for any neighbourhood relation F on X,

$$xE_{n_F}S \Leftrightarrow x \in n_F(S) \Leftrightarrow xFS.$$

This suggests $E_{n_F} = F$. Hence, we have explicitly constructed a bijection between $\mathbf{Nb}_X$ and $\mathbf{CABAO}^{\mathrm{op}}_X$, and this completes the proof that $(-)^+ : \mathbf{Nb} \to \mathbf{CABAO}^{\mathrm{op}}_X$ gives us a concrete isomorphism. ∎

It is precisely in the sense of Proposition 3.10 that we say, neighbourhood frames are the most general type of semantics for the basic modal logic language $\mathcal{L}$, or $\mathcal{L}_\Sigma$ for any indexed set Σ, because **Nb** is directly dual to the algebraic description **CABAO**.

The above defined semantic functor on **Nb** obviously restricts to one on monotone neighbourhood frames,

$$(-)^+ : \mathbf{Mon} \to \mathbf{CABAO}^{\mathrm{op}}.$$

Let $\mathcal{M}$ be the class of *monotone operators*, we further show that **Mon** are precisely dual to $\mathbf{CABAO}^{\mathrm{op}}_{\mathcal{M}}$:

Proposition 3.11: The semantic functor on **Mon** restricts to $\mathbf{CABAO}^{\mathrm{op}}_{\mathcal{M}}$, and it consists of a concrete isomorphism,

$$(-)^+ : \mathbf{Mon} \cong \mathbf{CABAO}^{\mathrm{op}}_{\mathcal{M}},$$

Proof Using the duality in Proposition 3.10, we only need to show that for any neighbourhood frame (X, E), n_E is monotone iff E is monotone as a neighbourhood relation, since any operator on $\wp(X)$ is exactly n_E of some neighbourhood relation E. Now for any $S \subseteq T \subseteq X$, we have

$$n_E(S) \subseteq n_E(T) \Leftrightarrow \forall x \in X[x \in n_E(S) \Rightarrow x \in n_E(T)] \Leftrightarrow \forall x \in X[xES \Rightarrow xET].$$

The final condition exactly expresses the fact that E is a monotone neighbourhood frame. This completes the proof. ∎

In particular, since all the semantic functors of **Kr**, **Pre**, **Eqv** and **Top** all land in $\mathbf{CABAO}^{\mathrm{op}}_{\mathcal{M}}$, it follows that all these categories can be fully concretely embedded in **Mon**.

The bounded morphisms in this case, viz. the morphisms that correspond to algebraic morphisms between CABAOs, is easy to characterise. We left the readers to see that f^{-1} is an algebraic morphism between the two CABAOs, iff for any $x \in X$ and $T \subseteq Y$, $fxFT \Leftrightarrow xFf^{-1}(T)$.

Remark 3.3: We can look at the relation between **Mon** and **Evi** again, in the context of semantic functors. Recall from Remark 2.2 that the category **Evi** of evidence spaces have the same objects as **Nb**, but with a different notion of morphisms. We have acutally showed there that **Mon** and **Evi** are concretely equivalent as categories. The equivalence we constructed for them is a functor $\mathbf{Evi} \to \mathbf{Mon}$, sending each evidence space (X, E) to (X, E^*), where E^* is the smallest monotone neighbourhood frame generated by E. This would directly induce a semantic functor for **Evi**, such that for any evidence space (X, E), the associated operator e_E on $\wp(X)$ is given by

$$E \mapsto E^* \mapsto n_{E^*} =: e_E.$$

Explicitly, for any $x \in X$ and any $S \subseteq X$, we have

$$x \in e_E(S) \Leftrightarrow xE^*S \Leftrightarrow \exists T[xET \,\&\, T \subseteq S],$$

which is exactly the semantics for evidence logic described in[5]. This has further confirmed our point in Remark 2.2 that, working with evidence spaces is the same as working with monotone neighbourhood frames, at least with respect to basic modal logic. ◀

3.1.4 Other Modalities and Semantic Functors

It is perhaps worth noticing that, the semantic functor is not uniquely determined by a topological category $(\mathcal{A}, |-|)$. It is certainly possible that, for the same topological category, it admits different ways of associating modalities. We will discuss some examples of this kind in this subsection.

There are one general type of examples that are universally applicable to all topological categories, inducing the same operators on each fibre. We call such constructions constant semantic functors:

Definition 3.3: A *constant semantic functor* is a concrete functor

$$U : \mathbf{Set} \to \mathbf{CABAO}^{\mathrm{op}},$$

where **Set** is considered to have the concrete structure $1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$. In other words, U is a section of the forgetful functor from $\mathbf{CABAO}^{\mathrm{op}}$ to **Set**.

Now from Example 2.1 (i). we know that $1_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ is the terminal object among all concrete categories. Hence, given any constant semantic functor U, for any construct we can form the following composition,

$$\mathcal{A} \xrightarrow{|-|} \mathbf{Set} \xrightarrow{U} \mathbf{CABAO}^{\mathrm{op}}$$

which would result in a semantic functor for $\mathcal{A}$. However, notice that for an arbitrary object in the fibre $\mathcal{A}_X$, the operator on $\wp(X)$ induced by the above composition is always given by UX, which is the same for any A in $\mathcal{A}_X$. This also explains our terminology for constant semantic functors.

Example 3.1 (Constant Semantic Functors): Here we present some examples of constant semantic functors:

(i). We first have two trivial examples

$$\mathsf{T}_{(-)}, \mathsf{F}_{(-)} : \mathbf{Set} \to \mathbf{CABAO}^{\mathrm{op}},$$

such that T_X is the operator on $\wp(X)$ sending each subset S to the total set X, and F_X is the one sending each S to the empty set $\varnothing$. Functoriality is trivial in these two cases. In fact, for any function $f : X \to Y$, we actually have two algebraic morphisms,

$$f^{-1} \circ \mathsf{T}_Y = \mathsf{T}_X \circ f^{-1}, \quad f^{-1} \circ \mathsf{F}_Y = \mathsf{F}_X \circ f^{-1}.$$

Notice that T and F are dual to each other, in the sense we have described in the first part of Section 3.1.

(ii). A perhaps less trivial example is the constant semantic functor for *universal modalities*. Explicitly, it is a functor $u_{(-)} : \mathbf{Set} \to \mathbf{CABAO}^{\mathrm{op}}$, sending each set X to the universal modality u_X on $\wp(X)$, such that for any $S \subseteq X$,

$$u_X(S) = \begin{cases} X & S = X, \\ \varnothing & \text{otherwise.} \end{cases}$$

It is obviously functorial: For any function $f : X \to Y$ and any subset $S \subseteq Y$, we obviously have

$$f^{-1}u_Y(S) \subseteq u_X f^{-1}(S),$$

this is because f^{-1} preserves empty set and total set. For any construct $\mathcal{A}$, the composition of its forgetful functor $|-|$ with $u_{(-)}$ would send each object A in the fibre $\mathcal{A}_X$ to the universal operator u_X on $\wp(X)$. Though somewhat trivial, universal modalities are useful in many different context of modal logic.

(iii). We also have the constant semantic functor of *existential modalities*, dual to universal ones. It is a functor $e_{(-)} : \mathbf{Set} \to \mathbf{CABAO}^{\mathrm{op}}$, such that for any set X and any $S \subseteq X$,

$$e_X(S) = \begin{cases} \varnothing & S = \varnothing, \\ X & \text{otherwise.} \end{cases}$$

Similar reasoning as above can show the functoriality of $e_{(-)}$. The existential modality e_X is dual to the universal modality u_X, in the sense that for any subset S, as long as it is not the empty set, $e_X(S)$ would be the total set.

◀

As we have mentioned, these examples of constant semantic functors work for any construct $\mathcal{A}$, since **Set** is the terminal object in $\mathfrak{Conc}$, the category of concrete categories and concrete functors. Thus, in particular, these constructions can be applied to the previous examples we have discussed in this section, resulting in different semantic functors for them.

Less trivially, the same semantic model might both support epistemological and doxastic reasoning, which would result in different modalities modelling knowledge and belief of an agent. Here we provide one canonical example of this kind based on **Mon**, which is borrowed from the discussion of belief in evidence spaces presented in[5]:

Example 3.2 (Doxastic Modality on **Mon**): For any monotone neighbourhood frame (X, E), and any $x \in X$, we define an *x-scenario* as a maximal collection $\mathcal{X}$ of subsets in $E[x]$, such that it is finitely jointly consistent, i.e. for any $S_1, \cdots, S_n \in \mathcal{X}$, the intersection $\bigcap_{i=1}^n S_i$ is non-empty. We then define a new operator m_B on $\wp(X)$ as follows:

$$x \in m_B(S) \Leftrightarrow \text{there exists an } x\text{-scenario } \mathcal{X}, \exists v \in \bigcap \mathcal{X}, v \in S.$$

Intuitively, this operator models the possibility of belief, i.e. for some maximally consistent collection of evidences, there are some state satisfying all these evidences which supports the proposition. We restrict ourselves to the class of *flat* neighbourhood frames, in which any scenario has non-empty intersection. Evidently, every finite monotone neighbourhood frame is flat, and we use **FMon** to denote the full subcategory of flat monotone neighbourhood frames. We show that the above definition of the operator modelling the possibility of belief indeed provides a semantic functor on **FMon**.

Suppose we have a morphism $f : (X, E) \to (Y, F)$ between two monotone neighbourhood frames, and suppose that (X, E) is flat. Let m_B, n_B be the two modalities on (X, E), (Y, F), respectively, according to our above description. We verify that f^{-1} is continuous for them. For any $x \in X$, $T \subseteq Y$, suppose we have $fx \in n_B(T)$, which means that there is a fx-scenario $\mathcal{Y}$ in Y, such that $T \cap \bigcap \mathcal{Y}$ is non-empty. Now consider the collection of subsets as follows,

$$f^{-1}\mathcal{Y} := \{ f^{-1}(S) \mid S \in \mathcal{Y} \}.$$

This would be a finitely consistent collection of subsets in X, since f^{-1} preserves arbitrary intersection. By Zorn's Lemma, it could be extended to a maximally finitely consistent family $\mathcal{X}$, and by flatness, its intersection is non-empty. This in particular means that $f^{-1}T \cap \bigcap \mathcal{X}$ is non-empty, since $f^{-1}T \cap f^{-1}\mathcal{Y}$ is non-empty, $\mathcal{X}$ is non-empty and $\mathcal{X}$ extends $f^{-1}\mathcal{Y}$. This shows that there is a well-defined semantic functor

$$(-)^+_B : \mathbf{FMon} \to \mathbf{CABAO}^{\mathrm{op}},$$

which supports the notion of belief in flat monotone neighbourhood frames. ◀

3.1.5 Coalgebraic Modal Logic and Topological Structures

At the end of this section, we briefly indicate how the other categorical generalisation of modal logic, viz. the coalgebraic approach of modal logic, can be treated in our general framework of topological categories and semantic functors. This topic does not fall under the main focus of this thesis, so we will not treat it in its full depth. However, it is conceptually important to see that, the coalgebraic approach of modal logic, which is well-acknowledged as one of the most general approach of modal logic, can be treated in our framework.

There are many important variations of coalgebraic modal logic. For simplicity, here we describe a version which is a slight variant of Pattinson's approach outlined in[49], since it simultaneously general enough to accommodate all the examples we care about, yet bares formal resemblance of the modal language we use in our work. For more general approaches of coalgebraic modal logic, see the relevant chapters in[35].

Let β be an endo-functor on **Set**. Recall a β-coalgebra based on a set X is simply a function $X \to \beta X$. What's crucial to Pattinson's approach is the following notion of a predicate lifting:

Definition 3.4: Let $\wp : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ be the *contravariant* power set functor. A *predicate lifting* of the endo-functor β is a natural transformation

$$\mu : \wp \circ \beta \to \beta.$$

Explicitly, it is a collection of functions $\mu_X : \wp(X) \to \wp(\beta X)$ for any set X, such that for any $f : X \to Y$ the following diagramme commute,①

$$\begin{array}{ccc} \wp(X) & \xrightarrow{\mu_X} & \wp(\beta X) \\ {\scriptstyle f^{-1}}\uparrow & & \uparrow{\scriptstyle (\beta f)^{-1}} \\ \wp(Y) & \xrightarrow[\mu_Y]{} & \wp(\beta Y) \end{array}$$

When an endo-functor β is equipped with a predicate lifting μ, such data could be used to build coalgebraic semantics for modal logic. First, we observe that for any β-coalgebra $R : X \to \beta X$, we can associate it with an operator on $\wp(X)$ defined as the following composition,

$$\wp(X) \xrightarrow{\mu_X} \wp(\beta X) \xrightarrow{R^{-1}} \wp(X).$$

We denote this operator as m_R. This way, any β-coalgebra $R : X \to \beta X$ induces a CABAO of the form $(\wp(X), m_R)$. To extend this into a semantic functor, we first need to study what are the appropriate morphisms between coalgebras.

The usual definition of morphisms between coalgebras it as follows. Given two β-coalgebras $R : X \to \beta X$ and $Q : Y \to \beta Y$, a coalgebra morphism between them is a

① Notice that, according to our definition, $\wp$ is a functor from **Set** to $\mathbf{Set}^{\mathrm{op}}$, hence a natural transformation $\mu : \wp\circ\beta \to \wp$ consists of a morphism $\mu_X : \wp(\beta X) \to \wp(X)$ *in* $\mathbf{Set}^{\mathrm{op}}$, which is an ordinary function $\mu_X : \wp(X) \to \wp(\beta X)$!

function $f : X \to Y$, which makes the following diagramme commute,

$$\begin{array}{ccc} X & \xrightarrow{R} & \beta X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \beta f} \\ Y & \xrightarrow[Q]{} & \beta Y \end{array}$$

However, it is well-known that if we have a coalgebra morphism, then the resulting semantics of the two coalgebraic models are equivalent for the modal language. In other words, they correspond to algebraic morphisms between the induced CABAOs:

Proposition 3.12: Let $R : X \to \beta X$ and $Q : Y \to \beta Y$ be any two β-coalgebras. For any coalgebra morphism $f : X \to Y$ between them, f^{-1} will be an algebraic morphism for the two induced operators m_R, m_Q.

Proof By definition of the induced operators on $\wp(X)$ and $\wp(Y)$, f^{-1} is an algebraic morphism iff the outer square in the following diagramme commutes,

$$\begin{array}{ccccc} \wp(X) & \xrightarrow{\mu_X} & \wp(\beta X) & \xrightarrow{R^{-1}} & \wp(X) \\ \uparrow{\scriptstyle f^{-1}} & & \uparrow{\scriptstyle (\beta f)^{-1}} & & \uparrow{\scriptstyle f^{-1}} \\ \wp(Y) & \xrightarrow[\mu_Y]{} & \wp(\beta Y) & \xrightarrow[Q^{-1}]{} & \wp(Y) \end{array}$$

The left square commutes due to naturality of μ; the commutativity of the right square can be obtained by applying the contravariant power set functor $\wp$ to the commutative square in the definition of f being a coalgebra morphism. Hence, f^{-1} is indeed an algebraic morphism for m_R, m_Q. ∎

Now let $\mathbf{Set}_\beta$ to denote the category of β-coalgebras with coalgebra morphisms between them. For any predicate lifting μ, Proposition 3.12 shows that it would induce a semantic functor of the type

$$(-)^+_\mu : \mathbf{Set}_\beta \to \mathbf{CABAO}^{\mathrm{op}},$$

which actually lands in algebaic morphisms between CABAOs. Many coalgebraic semantics for modal logic arise this way. Here for the limited space, we only show how the coalgebraic framework subsumes the most well-known Kripke semantics.

Example 3.3 (Coalgebraic Description of Kripke Frames): The endo-functor involved in the coalgebraic description of Kripke frames is the *covariant* power set functor $\mathcal{P}$, sending each set X to $\wp(X)$, and any function $f : X \to Y$ to $\exists_f : \wp(X) \to \wp(Y)$.

It is then easy to see that, the data of a $\mathcal{P}$-coalgebra $R : X \to \wp(X)$ is tantamount to give a relation on X, with the identification $xRy \Leftrightarrow y \in R(x)$. The predicate lifting μ of $\mathcal{P}$ on component X is the map $\mu_X : \wp(X) \to \wp(\wp(X))$, defined as follows,

$$S \mapsto \{ T \mid T \subseteq S \}.$$

Naturality of μ is easy to verify. Then for a $\mathcal{P}$-coalgebra $R : X \to \wp(X)$, the induced operator on $\wp(X)$ is given by

$$m_R(S) = R^{-1}(\mu_X(S)) = \{ x \mid R(x) \subseteq S \} = \forall_{R^\dagger} S,$$

which recovers the usual induced operator for Kripke frames. Now a $\mathcal{P}$-coalgebra morphism between $R : X \to \wp(X)$ and $Q : Y \to \wp(Y)$ is a function $f : X \to Y$, which satisfies $Q \circ f = \exists_f \circ R$. Expand this definition, it is precisely the p-morphisms we discussed before in Proposition 3.3. ◀

Remark 3.4: Here we have only treated the category of coalgebras $\mathbf{Set}_\beta$ as a construct, not a topological category. As we've mentioned earlier, to mere give interpretations of the basic modal formulas does not require the full structure of a topological category. However, they will be needed in future part of this chapter, when more features of the modal language are treated. Under most circumstances, when the endo-functor is well-behaved enough, the category $\mathbf{Set}_\beta$ would indeed be topological, but a full exploration on this line is beyond the topic of this thesis.

On a more conceptual level, it will be more and more clear through this chapter that, the coalgebraic approach of modal logic, and our use of topological categories, are *two separate directions* of possible extension of basic modal logics. For the former, the exact shape of the modal language is vastly generalised for different endo-functors, and the satisfaction relation is treated more like a bisimulation between the syntax and a coalgebra[50]. However, the topological category structure allows the generalisation of the syntactic features of modal logic in another way, as explained in the remaining sections of this chapter. ◀

3.2 Modal Strength and Dependence Atoms

In the previous section, we have mainly focused on how the modal language $\mathcal{L}$ with a single modality, can be interpreted in topological categories, or more generally, concrete categories, with a semantic functor. In this section, we will proceed to study how multi-agent case could be treated, and how to compare the modal strength of different modalities, in a both semantic and syntactic level.

Again let $(\mathcal{A}, |-|)$ be a topological category, or more generally, a concrete category, with a semantic functor $(-)^+$. Now in the multi-agent case, when the indexed set Σ is not a singleton, generally we need to work in the induced category $(\mathcal{A}^\Sigma, |-|^\Sigma)$ to provide the semantics of $\mathcal{L}_\Sigma$. This is simple, since the concrete functor $(-)^+$ would also induce a concrete functor of the following type, which we also denote as $(-)^+$,

$$(-)^+ : (\mathcal{A}^\Sigma, |-|^\Sigma) \to (\mathbf{CABAO}^\Sigma)^{\mathrm{op}}.$$

Given a tuple $(A_a)_{a\in\Sigma}$, we simply define

$$(A_a)^+_{a\in\Sigma} = (A^+_a)_{a\in\Sigma}.$$

The index structure $(A^+_a)_{a\in\Sigma}$ then provides the interpretation of each modality $\Box_a$ in the language $\mathcal{L}_\Sigma$ for any $a \in \Sigma$. Intuitively, different agents correspond to different objects in the same fibre.

For any set X and any interpretation function $V : \mathsf{P} \to \wp(X)$ on X, all the $\mathcal{A}$-objects in the fibre $\mathcal{A}_X$ will induce an interpretation for any formula φ in $\mathcal{L}$ as subsets of X. Then a natural question is, how these different interpretations in a fibre relates to each other?

It turns out, the relation in the fibre $\mathcal{A}_X$ also signifies their *modal strength* in some sense. Explicitly, suppose we have two $\mathcal{A}$-objects A, B in the fibre $\mathcal{A}_X$, and $A \leq B$. The functor $(-)^+$ then gives us two objects A^+, B^+. In particular, they are the complete atomic Boolean algebra $\wp(X)$ equipped with operators m_A, m_B, respectively. Since $(-)^+$ is a concrete functor, the following identity function must be a morphism in **CABAO**,

$$1_X : \left|B^+\right| \to \left|A^+\right|.$$

In particular, according to our definition of morphisms in **CABAO**, this means we have

$$m_B \subseteq m_A,$$

in the sense that $m_B(S) \subseteq m_A(S)$ for any $S \subseteq X$. When we read the modal operations as epistemic notions, like "agent knows" or "agent believes", this suggests that there is an *dependence* between the two agents' knowledge or beliefs, i.e. whenever B knows something, A also knows, where the two agents are represented by A, B.

Perhaps more fundamental is the following *local* notion of dependence order between operators:

Definition 3.5: For any two operators m, m' on the CABA $\wp(X)$, and for any $U \subseteq X$, we say m' *locally depends on* m, denoted as $m \subseteq_U m'$, if for any $S \subseteq X$ and any $x \in U$,

$$x \in m(S) \Rightarrow x \in m'(S).$$

In other worlds, $m \subseteq_U m'$ iff $U \cap m(-) \subseteq U \cap m'(-)$ globally. When U is taken to be a singleton $\{x\}$, the condition is simplified as follows,

$$m \subseteq_x m' \Leftrightarrow \forall S \subseteq X[x \in m(S) \Rightarrow x \in m'(S)].$$

When this happens, we also say that m' depends on m locally at x.

From Definition 3.5, it is trivial to observe the following simple result:

Lemma 3.4: For any operators m, m' on $\wp(X)$, there is maximal subset U of X that $m \subseteq_U m'$ holds.

Proof By definition, for the empty set $\varnothing$ we always have $m \subseteq_\varnothing m'$, since the universal quantification $\forall x \in \varnothing$ is true vacuously. Furthermore, local dependence is closed under taking unions, since it is trivial to note that

$$m \subseteq_{\bigcup_{i \in I} U_i} m' \Leftrightarrow \forall i \in I . m \subseteq_{U_i} m'.$$

Thus, we can simply define the maximal subset U as follows,

$$U = \bigcup \{ V \mid m \subseteq_V m' \} = \{ x \mid m \subseteq_x m' \}. \qquad \blacksquare$$

These observations suggest that we can extend our language $\mathcal{L}_\Sigma$ to allow for dependence atoms, as introduced in[6-7]. Explicitly, we add atomic propositions of the form $K_a b$ for any $a, b \in \Sigma$ into our language, and we denote this extended language as $\mathcal{L}_\Sigma^D$. To define the interpretation of this extended language $\mathcal{L}_\Sigma^D$ within the construct $\mathcal{A}$ with the semantic functor $(-)^+$, the only additional clause we need to specify for the language is the interpretation of the dependence atoms. Suppose we have specified a Σ-index object $(A_a)_{a \in \Sigma}$ in $\mathcal{A}^\Sigma$ over the set X, and through the functor $(-)^+$ suppose they corresponds to a family of operators $(m_a)_{a\in\Sigma}$ on $\wp(X)$. The interpretation of $K_a b$ for any $a, b \in \Sigma$, is then given by the following clause,

$$\begin{aligned} [\![K_a b]\!]_{(A_a)_{a\in\Sigma}} &= \text{ the maximal subset } U \text{ of } X \text{ that } m_b \subseteq_U m_a \text{ holds;} \\ &= \{\, x \mid m_b \subseteq_x m_a \,\}. \end{aligned}$$

Now as usual, we can define the local satisfaction relation as follows,

$$(A_a)_{a\in\Sigma}, x \vDash K_a b \Leftrightarrow x \in [\![K_a b]\!]_{(A_a)_{a\in\Sigma}} \Leftrightarrow m_b \subseteq_x m_a.$$

In this sense, the interpretation $[\![K_a b]\!]$ of the dependence atom is simply the set of all points where it locally holds. One thing to notice is that, the interpretation of the dependence atoms is independent of the choice of interpretation function V of other propositional variables. We may simply add the above clause into the truth definition, and we will have a complete description of the interpretation of formulas in $\mathcal{L}^D_\Sigma$.

When interpreting the modalities as knowledge of belief, we not only want to talk about *external*, or *actual*, dependence among agents, like we have done above; we would also like to study *knowable or believable dependence* between agents internally. In these cases, we will assume that our semantics based on the construct $\mathcal{A}$ and the semantic functor $(-)^+$ will support at least the **S4** fragment, i.e. the semantic functor is required to restrict to the following type,

$$(-)^+ : \mathcal{A} \to \mathbf{CABAO}^{\mathrm{op}}_{\mathcal{M}},$$

where $\mathcal{M}$ is a class of operators that are at least decreasing. Both the semantic functors constructed for **Pre**, **Eqv** and **Top** in Section 3.1 are of this type. In this case, knowable or believable dependence is simply the composite formula $\Box_a K_a b$, with the dependence atom $K_a b$ modalised by $\Box_a$; and since the operators involved are decreasing, knowable dependence implies dependence locally, in the sense that the following always hold,

$$[\![\Box_a K_a b]\!] \subseteq [\![K_a b]\!].$$

Example 3.4 (Dependence in **Kr**, **Pre** and **Eqv**): Let's first look at the interpretation of dependence and knowable dependence in the Kripkean relational semantics. Let Σ be our indexed set, and suppose we have a Σ-indexed family of relations $(R_a)_{a\in\Sigma}$ on the set X, or equivalently an object in $\mathbf{Kr}^\Sigma$ over X. From Definition 3.5, for any $x \in X$,

$x \in [\![K_a b]\!]$ iff the following holds,

$$\forall S \subseteq X[x \in \forall_{R_a^\dagger}(S) \Rightarrow x \in \forall_{R_b^\dagger}(S)].$$

By the adjunction in Lemma 3.2 and the fact that $\exists_R(\{x\}) = R[x]$, this condition is equivalent to

$$\forall S \subseteq X[R_a[x] \subseteq S \Rightarrow R_b[x] \subseteq S],$$

which simply says that $R_a[x] \subseteq R_b[x]$. In other words, in the relational context, we can write the semantic condition for the dependence atoms locally as follows,

$$(X, (R_a)_{a \in \Sigma}), x \vDash K_a b \Leftrightarrow \forall y[x R_a y \Rightarrow x R_b y].$$

This is an extremely natural semantic definition for local dependence in relational frameworks. If $K_a b$ locally holds at x, then at this point x the set of possibilities acknowledged by agent a is less than that of agent b, hence whatever b knows or believes, a also does.

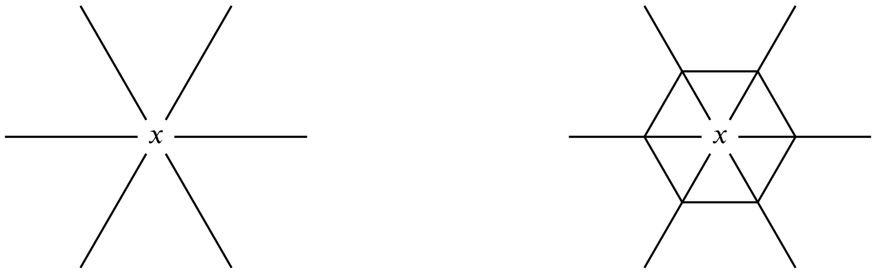
The modalised dependence atom $\Box_a K_a b$ does not make much sense in the most general relational frames **Kr**, since $\forall_{R^\dagger}$ is not necessarily decreasing for arbitrary relation R. In fact, $\Box_a K_a b$ could locally hold at a point x simply due to the fact that there is no points R_a-related to x, which has nothing to do with dependence at all. However, as we've mentioned, knowable dependence $\Box_a K_a b$ can be fruitfully considered in both **Pre** and **Eqv**.

For preorders, $\Box_a K_a b$ locally holds at x means that, not only we have $R_a[x] \subseteq R_b[x]$, but also for any point y emanates from x along R_a, $R_a[y] \subseteq R_b[y]$. In particular, we have the following simple result:

Fact 3.1: For any $x \in X$, $\Box_a K_a b$ locally holds at x if for any $y, z \in R_a[x]$, if $y R_a z$ then we also have $y R_b z$.

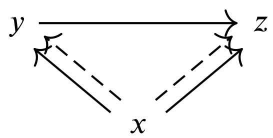
If we fix y as x itself, we recover the clause for $K_a b$ locally holds at x, but they are not equivalent. Intuitively, to test whether $K_a b$ locally holds at x or not, is to compare the *skeleton* emanates from x by R_a and R_b, while to test $\Box_a K_a b$ is to further compare the *net* emanates from x by the two relations, as indicated in the following figure:

For example, let the solid and dashed lines in the following diagramme represent relations for agent a, b, respectively. Then in the following preorder, $K_a b$ locally holds



(a) The skeleton emanated from x (b) The net emanates from x

at x, but $\Box_a K_a b$ fails to hold:



Interestingly, in **Eqv** the two notions actually come to one. Let R_a, R_b be two equivalence relation on X. For any $x \in X$, we have $R_a[x] \subseteq R_b[x]$ iff the R_a-equivalence class of x is smaller than the R_b-equivalence class of x. This, of course, uniformly holds for any y in the same R_a-equivalence class of x. Thus, we have that when all the relations are equivalence relations,

$$[\![K_a b]\!] = [\![\Box_a K_a b]\!].$$

Since the majority approaches in the literature that uses relational frames to model knowledge would use equivalence relations, the difference of dependence and knowable dependence is not present to them. ◀

Example 3.5 (Dependence in **Top**): Next, we can also look at how dependence instantiates in the context of topological spaces **Top**. The study of dependence atoms in the way described above actually originated from the two sequential papers[6-7], generally in a topological setting. There, the two authors have developed the logic of functional dependence, or LFD, and the logic of continuous dependence, or LCD, in which case the latter subsumes the former. Hence, we will compare the more general description of dependence atoms, instantiated to the category **Top** of topological spaces, with the LCD approach.

Let $a, b \in \Sigma$ be two arbitrary agents, and let τ_a, τ_b on X be the induced topology on X of a and b, which by the semantic functor we described in Section 3.1.2 correspond to the two topological interior operators j_a, j_b on $\wp(X)$. Again from our general formulation, for any $x \in X$ we have $K_a b$ locally holds at x, if, and only if, the following holds,

$$\forall S \subseteq X[x \in j_b(S) \Rightarrow x \in j_a(S)].$$

For a more concrete description in terms of topological spaces, we observe the following simple equivalence:

Lemma 3.5: For any $x \in X$ the following two conditions are equivalent:

1. $K_a b$ locally holds at x;
2. For any b-open neighbourhood U of x, there exists an a-open neighbourhood V of x contained in U.

Proof (1) $\Rightarrow$ (2): Suppose we have a b-open neighbourhood U of x, then $x \in j_b(U)$ since $j_b(U) = U$. By the fact that $K_a b$ locally holds at x we also have $x \in j_a(U)$. It then follows that there is a smaller a-open set $j_a(U) \subseteq U$ that is also a neighbourhood of x.

(2) $\Rightarrow$ (1): Given any subset S, suppose $x \in j_b(S)$. Then $j_b(S)$ is a b-open neighbourhood of x, thus there exists an a-open neighbourhood of x contained in $j_b(S)$ as well. This implies $x \in j_a(S)$. ∎

Lemma 3.5 then implies that, our general description of local dependence in the context of topological spaces, corresponds exactly to what in[6] called *conditional knowability*. This also implies that the knowable dependence $\Box_a Kab$ discussed here is the same notion as what in[6] called *epistemic dependence*. As one can see, epistemic dependence, or knowable dependence, is definable using local dependence. Hence in this sense, $K_a b$ as we described here is the most fundamental notion. However, it is not the main dependence notion analysed in[6]. The two authors have several reasons for using epistemic dependence, rather than the more fundamental conditional knowability, as the primary notion — some conceptual, others technical; for more details of this comparison, see the discussions there.① ◀

Example 3.6 (Dependence in Nb and Mon): Finally, we briefly discuss what is the concrete meaning of dependence atoms in neighbourhood frames. Given two evidence relation $E_a, E_b \subseteq X \times \wp(X)$ on X corresponding to the two agents $a, b \in \Sigma$, recall that

① There is a conflict of notations between ours and the ones used in[6]. The weaker notion of conditional knowability, as we write as $K_a b$ here, are actually denoted as $k_a b$; what they call $K_a b$ is the knowable counterpart of the dependence atom, viz. what would be denoted as $\Box_a K_a b$ in our paper. *Caveat lector!*

the induced operators on $\wp(X)$ are defined as follows: For any $S \subseteq X$,

$$n_a(S) = \{ x \mid xE_aS \}, \quad n_b(S) = \{ x \mid xE_bS \}.$$

Then again, from the general description given in Definition 3.5, in this context for any $x \in X$, $x \in [\![K_a b]\!]$ iff the following holds,

$$\forall S \subseteq X[x \in n_b(S) \Rightarrow x \in n_a(S)].$$

Equivalently, purely in terms of the evidence relations, it means that

$$x \in [\![K_a b]\!] \Leftrightarrow E_b[x] \subseteq E_a[x].$$

Again, this is an extremely natural condition for dependence in evidence frames. If we think of $E_a[x]$ and $E_b[x]$ as the evidence of the two agents a, b at stage x, then b depends on a locally at x means exactly that, the evidence set of b is smaller than the evidence set of a. However, since in general the evidence logic is not decreasing, it does not make much sense to describe the modalised dependence atoms in this case. When evidence frames are restricted to smaller classes which do result in decreasing operators, it would be interesting to describe what are the corresponding notion of knowable dependence. However, due to limited space, a full analysis of this is beyond the topic of this thesis. ◀

3.3 Group Knowledge and Topological Fibres

In the previous two sections, we have studied how the modal logic of individual agents could have arisen from topological categories, or in fact, concrete categories, $\mathcal{A}$ and semantic functors $(-)^+$, and how can we compare the modal strength of two agents in terms of local dependence atoms and knowable dependence. However, in logic we not only need to model the inference of different agents individually, but also we would like to study how *a group of agents*, as a whole, reasons. From a mathematical perspective, this corresponds to the transformation of an object in $\mathcal{A}^\Sigma$, viz. Σ-indexed objects $(A_a)_{a\in\Sigma}$ over the same set X in $\mathcal{A}$, to a single object in $\mathcal{A}$ over the same set X, corresponding to the single agent representing the group formed by Σ. Hence, we will in general be interested in concrete functors of the form

$$\mathcal{A}^\Sigma \to \mathcal{A}.$$

It turns out, in this case, that it is quite important for $\mathcal{A}$ to be a topological category, rather than a mere construct, to describe the most commonly seen such functors in the literature. Hence, we will assume $\mathcal{A}$ is a topological category equipped with a semantic functor $(-)^+$ in this section, which, strictly speaking, is not required in the previous two sections.

Notice that, an object $(A_a)_{a\in\Sigma}$ over X can be though of Σ-indexed family of objects in $\mathcal{A}_X$, and $\mathcal{A}_X$ is a complete lattice, according to Corollary 2.1. There are two *canonical* ways to do this in general — canonical in the sense that the description is independent from the choice of Σ — indicated by the following two concrete functors,

$$\bigwedge, \bigvee : (\mathcal{A}^\Sigma, |-|^\Sigma) \to (\mathcal{A}, |-|).$$

As the above symbols suggest, for any tuple $(A_a)_{a\in\Sigma}$ in $\mathcal{A}^\Sigma$, the above two functors act on objects as follows,

$$\bigwedge(A_a)_{a\in\Sigma} = \bigwedge_{a\in\Sigma} A_a, \quad \bigvee(A_a)_{a\in\Sigma} = \bigvee_{a\in\Sigma} A_a.$$

Lemma 3.6: For any topological category $(\mathcal{A}, |-|)$, both of the following assignments define concrete functors from $\mathcal{A}^\Sigma$ to $\mathcal{A}$, for any indexed set Σ,

$$\bigwedge(A_a)_{a\in\Sigma} = \bigwedge_{a\in\Sigma} A_a, \quad \bigvee(A_a)_{a\in\Sigma} = \bigvee_{a\in\Sigma} A_a.$$

Proof Given an $\mathcal{A}^\Sigma$-morphism $f : \left|(A_a)_{a\in\Sigma}\right| \to \left|(B_a)_{a\in\Sigma}\right|$ in $\mathcal{A}^\Sigma$, by definition it means that for any $a \in \Sigma$, $f : \left|A_a\right| \to \left|B_a\right|$ is an $\mathcal{A}$-morphism. Now by the description of the fibre structure of topological categories in Section 2.4, this map considered as a function $f : \left|\bigwedge_{a\in\Sigma} A_a\right| \to \left|\bigwedge_{a\in\Sigma} B_a\right|$ is an $\mathcal{A}$-morphism, if the following holds,

$$\bigwedge_{a\in\Sigma} A_a \leq f^*\left(\bigwedge_{a\in\Sigma} B_a\right) = \bigwedge_{a\in\Sigma} f^* B_a.$$

Again, we have this equality because in a topological category we have $f_! \dashv f^*$, which means f^* is a right adjoint, and right adjoint preserves arbitrary meets. However, since $f : \left|A_a\right| \to \left|B_a\right|$ is an $\mathcal{A}$-morphism, which means for any $a \in \Sigma$ we have

$$A_a \leq f^* B_a.$$

This then implies the above inequality also holds, and by Lemma 2.1 it shows $\bigwedge$ is a well-defined functor.

Similarly, by the equivalent formulation involving $f_!$ to test whether a set map is an $\mathcal{A}$-morphism, and using the fact $f_!$ is a left adjoint hence preserves arbitrary joins, we can show $\bigvee$ is also a well-defined functor. ∎

Notice that, it is already evident from above proof, that the functoriality of $\bigwedge$ and $\bigvee$ is closely related to the fibre structure in topological categories. These two functors then allow us to combine a group of agents into a single one, and the two ways corresponds to two different readings what a group of agents is. We will denote them as the $\bigwedge$-combination or $\bigvee$-combination of group agents.

Intuitively, the $\bigwedge$-combination means the group *shares the information* of each individual. Let $(A_a)_{a\in\Sigma}$ be an $\mathcal{A}^\Sigma$-object over the set X, then for any $a \in \Sigma$, in the fibre $\mathcal{A}_X$ we obviously have

$$\bigwedge_{a\in\Sigma} A_a \leq A_a.$$

Since the semantic functor $(-)^+$ is a functor from $\mathcal{A}$ to the opposite category $\mathbf{CABAO}^{\mathrm{op}}$, it reverses the order in the fibres. If we use m_a to denote the associated operator of A_a on $\wp(X)$ for any $a \in \Sigma$, and use $m_\wedge$ to denote the one associated to $\bigwedge_{a\in\Sigma} A_a$, then it follows that

$$m_a \subseteq m_\wedge,\ \forall a \in \Sigma.$$

In particular, this means that whatever agent a knows or believes, so does the group, and this holds for any agent in this group; furthermore, the group knowledge, modeled by the operator $m_\wedge$, is the universal one that has this property. The group acts like an agent whose information is exactly the combination of all agents in this group. More informally, the group represented by $\bigwedge_{a\in\Sigma} A_a$ is like the case when each individual agent in the group has come to a single location, and put all of their information on the table where everyone could see. Hence, we also say that this type of group knowledge corresponds to *physically* putting the group together.

On the other hand, the $\bigvee$-combination means the group *shares the uncertainties* of each individual. Similar to above, if we let $m_\vee$ to denote the operator corresponds to $\bigvee_{a\in\Sigma} A_a$ along the semantic functor $(-)^+$, we would have that, for any $a \in \Sigma$,

$$m_\vee \subseteq m_a.$$

This means that for the combined group, if it knows or believes something, then necessarily each individual also knows or believes this, and the group agent is the universal one that has this property.

Furthermore, in most of the contexts where the modal operators have epistemic or doxastic readings, the semantic functor will restrict to the following type,

$$(-)^{+} : \mathcal{A} \to \mathbf{CABAO}_{M}^{\mathrm{op}},$$

where $\mathbf{CABAO}_M$ denotes the class of monotone and idempotent operators. In this case, more is true: For any $a_1, \cdots, a_n \in \Sigma$, we actually have

$$m_{\vee} \subseteq m_{a_1} \circ \cdots \circ m_{a_n}.$$

This is due to the fact that $m_{\vee} \subseteq a_i$ for each i, $m_{\vee} \circ m_{\vee} = m_{\vee}$, and that each operators is monotone, since the semantic functor, by assumption, always induces idempotent and monotone operators. In this case, what the combined group knows/believes is much more restrictive, in that if the group knows/believes something, then for any agents $a_1, \cdots, a_n$ in the group, we also have that a_1 knows/believes that a_2 knows/believes that $\cdots$ a_n knows/believes something. Informally, this is like treating all the agents in Σ *abstractly* as a group agent, not physically putting them together. The notion of knowledge or belief of this abstract group agent is much more stringent than requiring all of them knows or believes something.

Similar to the case of the notion of local dependence, the existence of the two functor $\bigwedge$, $\bigvee$ provides us ways to semantically speak about forming group agents and reason about their knowledge and belief, and we would also like to explicitly refer to such operations on a syntactic level, by extending the language of basic modal logic. Usually, in a specific logical system, we only consider one way of combining agents, either using $\bigwedge$ and adopting this physical combination interpretation, or using $\bigvee$ and adopting the more abstract interpretation of groups. Also, by adding the ability of forming groups into the language, we also add some canonical dependence between different groups, given by which individuals they contains. For example, we have already mentioned that $m_a \subseteq m_{\wedge}$, for any $a \in \Sigma$, where $m_{\wedge}$ represents the $\bigwedge$-combination of the group of agents in Σ. Thus, for any individual $a \in \Sigma$, we would have that the knowledge/belief of a depends on the knowledge/belief of the $\bigwedge$-combined group. Below we give a much more systematic development.

For any indexed set Σ, we let Σ_l, Σ_r be synonyms for the power set $\wp(\Sigma)$. The language $\mathcal{L}^D_{\Sigma_l}$ and $\mathcal{L}^D_{\Sigma_r}$ are the modal languages with agent symbols in Σ_l, Σ_r, respectively, together with all the dependence atoms between these agents. Given an object $(A_a)_{a\in\Sigma}$ in $\mathcal{A}^\Sigma$ over the set X, we can interpret the modal operators for a group of agents in the two fragments as either $\bigwedge$- or $\bigvee$-combination of individual ones. For any subset $G \subseteq \Sigma$, we define the interpretation of $\Box_G$ in $\mathcal{L}^D_{\Sigma_l}$ as the operator on $\wp(X)$ associated to the object $\left(\bigwedge_{a\in G} A_a\right)^+$ in **CABAO**, where we are secretly identifying G as an indexed set, and using the following functor

$$\bigwedge : \mathcal{A}^G \to \mathcal{A}.$$

Similarly, for the language $\mathcal{L}^D_{\Sigma_r}$, $\Box_G$ is interpreted as the operator in $\left(\bigvee_{a\in G} A_a\right)^+$. Of course, for single agent a, viewed as a singleton $\{a\}$ in Σ_l or Σ_r, its interpretation under the two fragments coincide, which still corresponds to the usual interpretation of the operator in A_a^+.

Once we have identified how to interpret the modal operators of groups, we can simply treat them as single agents, and it is then natural to extend to the interpretation of dependence atoms between them. The upshot is that, under such an interpretation, we can identify the following valid logical axioms in the two languages $\mathcal{L}^D_{\Sigma_l}$ and $\mathcal{L}^D_{\Sigma_r}$:

Proposition 3.13: Given any semantic functor $(-)^+ : \mathcal{A} \to \mathbf{CABAO}^{\mathrm{op}}$, the following axioms are valid in $\mathcal{L}^D_{\Sigma_l}$:

- *Inclusion*: $K_G H$, provided $H \subseteq G$;
- *Additivity*: $K_G H \wedge K_G P \to K_G(H \cup P)$;
- *Transitivity*: $K_G H \wedge K_H P \to K_G P$;
- *Transfer*: $K_G H \wedge K_H \varphi \to K_G \varphi$.

Completely dually, the following axioms are valid in $\mathcal{L}^D_{\Sigma_r}$:

- *Inclusion*: $K_H G$, provided $H \subseteq G$;
- *Additivity*: $K_H G \wedge K_P G \to K_{H\cup P} G$;
- *Transitivity*: $K_G H \wedge K_H P \to K_G P$;
- *Transfer*: $K_G H \wedge K_H \varphi \to K_G \varphi$.

Proof We only prove the validity for the language $\mathcal{L}^D_{\Sigma_l}$; the other case is completely dual. Let $(A_a)_{a\in\Sigma}$ be any object in $\mathcal{A}^\Sigma$ over X, and let $(m_a)_{a\in\Sigma}$ be the corresponding operators on $\wp(X)$ under the semantic functor $(-)^+$. Notice that the functor $(-)^+$ is from $\mathcal{A}$ to the opposite category of **CABAO**. Whenever we have $H \subseteq G \subseteq \Sigma$, the fact that in the fibre

$\mathcal{A}_X$ we have

$$A_G = \bigwedge_{a \in G} A_a \subseteq \bigwedge_{a \in H} A_a = A_H,$$

implies that for the induced operators corresponding to G, H,

$$m_H \subseteq m_G.$$

Hence, according to our definition of the interpretation of the dependence atoms, we have $X \vDash K_G H$. This shows the Inclusion axiom holds.

For any two groups H, P, by definition we have

$$A_{H \cup P} = \bigwedge_{a \in H \cup P} A_a = A_H \wedge A_P.$$

Hence, we must have

$$m_H \cup m_P \subseteq m_{H \cup P}.$$

Now locally, suppose for some $x \in X$ we have $x \in [\![K_G H]\!]$ and $x \in [\![K_G P]\!]$. Then for any $S \subseteq X$,

$$x \in m_{H \cup P}(S) \Rightarrow x \in m_H(S) \cup m_P(S).$$

Either $x \in m_H(S)$ or $x \in m_P(S)$, we would have $x \in m_G(S)$, according to our assumption that $K_G H$ and $K_G P$ locally holds at x. Hence, the Additivity law also holds.

The Transitivity and Transfer axioms both trivially follows from our way of defining the interpretation of dependence atoms. ∎

Corollary 3.3: Inclusion, and Additivity holds also for knowable dependence when the semantic functor $(-)^+$ on $\mathcal{A}$ provides *normal* modal logic interpretations, i.e. it lands in the class of operators which preserves finite meets; if the interpretation is furthermore decreasing, then the Transitivity and Transfer axioms hold as well.

Proof First suppose the semantic functor $(-)^+$ provides normal modal interpretations. Inclusion holds for knowable dependence follows from the fact that, any operator preserves the top element, since they preserves finite, hence empty, meets. Hence, from the fact that $[\![K_G H]\!] = X$, it follows that

$$[\![\Box_G \, K_G H]\!] = m_G[\![K_G H]\!] = m_G(X) = X.$$

Since operators which preserves finite meets are necessarily monotone, it is also easy to see that, for any $x \in X$,

$$x \vDash \Box_G K_G H \wedge \Box_G K_G P \Leftrightarrow x \vDash \Box_G (K_G H \wedge K_G P)$$
$$\Rightarrow x \vDash \Box_G (K_G (H \cup P)).$$

The first equivalence uses the fact that $\Box_G$ is normal, and the second implication uses the fact that $\Box_G$ is monotone, and the fact that Additivity holds for the dependence atoms. This shows that Additivity holds for knowable dependence as well.

Now if the induced modality is furthermore decreasing, we then know that

$$x \vDash \Box_G K_G H \Rightarrow x \in m_G [\![K_G H]\!] \Rightarrow m_H \subseteq_x m_G.$$

Suppose then $x \vDash \Box_G K_H \wedge \Box_H K_H P$, it follows that

$$x \vDash \Box_H K_H P \Rightarrow x \in m_H [\![K_H P]\!] \Rightarrow x \in m_G [\![K_H P]\!].$$

The second implication follows from the fact that m_G locally depends on m_H at x. Hence finally,

$$x \in m_G [\![K_G H]\!] \wedge m_G [\![K_H P]\!] = m_G [\![K_G H \wedge K_H P]\!] \subseteq m_G [\![K_G P]\!].$$

The final inequality is due to the fact that Transitivity holds for local dependence, and the fact that m_G is monotone. Thus, Transitivity also holds for knowable dependence. The Transfer axiom can be proved in similar fashion. ∎

As previously, we end this section by discussing how the general account of forming groups and their dependence instantiates in the main concrete examples we have in mind:

Example 3.7 (Group Agents in **Kr**, **Pre** and **Eqv**): First, we look at how to combine agents into groups in relational frames. From our general discussion previously, this is closely related to the lattice structure of the fibres $\mathbf{Kr}_X$, $\mathbf{Pre}_X$ and $\mathbf{Eqv}_X$. Among those, the structure of $\mathbf{Kr}_X$ is the most straight forward one. The meets and joins are simply computed as intersection and union of relations. For the other two classes, meets are also computed by intersections of relations, since preorders and equivalence relations are closed under taking intersection. However, for the joins we need to take the corresponding closure of the union of relations. For a family $(R_i)_{i \in I}$ of relations, we will write

$$\bigvee_{i \in I} R_i = \left(\bigcup_{i \in I} R_i \right)^*,$$

where, depending on the context, $(-)^*$ will refer to the preorder or equivalence closure of a relation.

Let Σ be any indexed set, and let $(R_a)_{a\in\Sigma}$ in $\mathbf{Kr}_X^\Sigma$ be Σ-indexed relations over X, representing each individual agent $a \in \Sigma$. Now for any $G \subseteq \Sigma$, the $\bigwedge$- and $\bigvee$-combination of the group G in all **Kr**, **Pre**, **Eqv** is then given by

$$R_{G_\wedge} = \bigwedge_{a\in G} R_a = \bigcap_{a\in G} R_a, \quad R_{G_\vee} = \bigvee_{a\in G} R_a,$$

where the meets and joins are taken in the suitable fibre. Such a result corresponds well to our general account. The $\bigwedge$-combination takes all the information owned by each individual together and form the group knowledge, hence the set of uncertainties for the group consists of only those points that all the group members are uncertain about. This is the epistemological interpretation of taking the meets to form the group, and this is usually called the *distributed knowledge* of a group.

On the other hand, the $\bigvee$-combination makes agents in the group share uncertainties. Especially in the case of **Pre** and **Eqv**, the uncertainty relation of the group agent is the one that combines all the possible links travelled along zig-zag paths of any individual in the group. More formally, we can write the $\bigvee$-combination as follows in both **Pre** and **Eqv**,

$$\bigvee_{i\in G} R_i = \bigcup_{1\leq n\leq\omega} \underbrace{\left(\bigcup_{i\in G} R_i\right) \circ \cdots \circ \left(\bigcup_{i\in G} R_i\right)}_{n \text{ times}}.$$

This formula further shows us that, for the abstract group G, x is $\bigvee_{i\in G} R_i$-related to y iff there exists a finite path $x \rightsquigarrow x_1 \rightsquigarrow \cdots \rightsquigarrow x_n \rightsquigarrow y$ from x to y, where each individual path is R_i-related for some $i \in G$. Hence, $\bigvee$-combination of groups corresponds exactly to what is commonly referred to as *common knowledge* in epistemic logic, which is a much more stringent condition for a group to possess knowledge. For more discussions on distributive knowledge and common knowledge, and other types of knowledge of a group, see[9] and the reference there in.

Also, we know from Section 3.1.1, all semantics for modal logic based on relational frames are normal, and the one based on **Pre** and **Eqv** are furthermore extending **S4**. This in particular shows that, all the axioms about (knowable) dependence atoms described in Proposition 3.13 are valid for Kripkean models based on preorders and equivalence relations. We also know from Example 3.4 that knowable dependence is strictly stronger than dependence in **Pre**, while they coincide in **Eqv**. The same results carry over to the more general dependence between group agents. ◀

Example 3.8 (Group Agents in **Top**): Now in the context of topological spaces, recall from Example 2.3 that the order in $\mathbf{Top}_X$, for any set X, is the *reverse* one of the inclusion order of topologies on X, i.e. $\tau \leq \gamma$ iff $\gamma \subseteq \tau$. In other words, the relation in $\mathbf{Top}_X$ corresponds to the "coarser than" relation. Now given an Σ-indexed family of topologies $(\tau_a)_{a \in \Sigma}$ on X, the $\bigwedge$-combination of the group is given by

$$\bigwedge_{a \in \Sigma} \tau_a = \left(\bigcup_{a \in \Sigma} \tau_a \right)^* ,$$

where $\left(\bigcup_{a \in \Sigma} \tau_a \right)^*$, in this context, denotes the coarsest topology on X which contains $\bigcup_{a \in \Sigma} \tau_a$ as open sets. In particular, it is the topology generated by the subbase $\bigcup_{a \in \Sigma} \tau_a$. The reason that $\bigwedge$-combination of the group corresponds to taking the union of the set of open sets is exactly because $\tau \leq \gamma$ iff $\tau \supseteq \gamma$.

In this case, the $\bigvee$-combination of group agent has a much simpler description, given by simply taking the intersection of topologies,

$$\bigvee_{a \in \Sigma} \tau_a = \bigwedge_{a \in \Sigma} \tau_a.$$

This is because, the definition of topology is formulated by a certain closure property, and in general, the collection of structures identify by such a closure property must be closed under taking intersections.

The analysis of dependence between groups of agents is one of the central topic in both[7] and[6]. In fact, their approach fits exactly to our general account of these, in the concrete example of the category of topological spaces **Top**. For example, as we've already established in Lemma 3.5, the general notion of local dependence in our description corresponds exactly to the conditional knowability given in[6]; furthermore, the axiomatisation of (knowable) dependence atoms between groups discussed in Proposition 3.13 and Corollary 3.3 are exactly the ones given in[7] and[6]. The connection will perhaps be more evident in Section 3.4, where we provide a new account of agents and groups as morphisms in the topological category, which is much more similar to the one given in[6]. ◀

Example 3.9 (Group Agents in **Nb** and **Mon**): Finally, we look at the case in the category of neighbourhood frames **Nb**, and its full subcategory **Mon** of monotone neighbourhood frames. Similarly to the case of **Top**, from Example 2.3 we know that, the order in the fibre $\mathbf{Nb}_X$ for any set X is *reverse* to the inclusion order of neighbourhood relations, when we view any neighbourhood relation E on X as a subset of $X \times \wp(X)$. However, since no further requirements are placed on neighbourhood relations in this most general case, for any Σ-indexed neighbourhood relations $(E_a)_{a\in\Sigma}$ on X, we then have

$$\bigwedge_{a\in\Sigma} E_a = \bigcup_{a\in\Sigma} E_a, \quad \bigvee_{a\in\Sigma} E_a = \bigcap_{a\in\Sigma} E_a.$$

Again, if we borrow the intuition from evidence logic, and think of neighbourhoods of a state x as its collection of evidence, this corresponds to our intuitive understanding of $\bigwedge$- and $\bigvee$-combination of agents: The $\bigwedge$-combination takes the group together physically, by combining every piece of evidence owned by each agent; while the $\bigvee$-combination is more prudent, which only takes the common evidence shared by the group.

What's more interesting is the case for monotone neighbourhood frames. Unlike the case of **Pre** and **Eqv** as for **Kr**, the monotone neighbourhood frames is closed both under arbitrary intersections and arbitrary unions, hence the $\bigwedge$- and $\bigvee$-combinations of groups in **Mon** is simply given by intersections and unions as well,

$$\bigwedge_{a\in\Sigma} E_a = \bigcup_{a\in\Sigma} E_a, \bigvee_{a\in\Sigma} E_a = \bigcap_{a\in\Sigma} E_a,$$

which means we do not need to further take the monotone-closure of a neighbourhood frame, as we have described in Remark 2.2. ◀

We end this section by two more remarks:

Remark 3.5: Here we briefly touch the possibility of adopting a much more general approach for studying group knowledge. Recall that, in the previous development of forming group knowledge, technically what we require is a natural family of functors $\mathcal{A}^\Sigma \to \mathcal{A}$, for any arbitrary indexed set Σ, with naturality to be understood in a categorical sense. The two types of functors $\bigwedge, \bigvee$ are only two possibilities of such sort, and we may find new interesting instances of such functors to support other notions of forming group knowledge. We leave this for the interested readers. ◀

Remark 3.6: After this section, some may wonder why we do not define a total language that allows forming both types of group knowledge, and consider what it would be like to reason in that language. This can be done in principle, but with a little estimating in advance one would find that the complexity of the resulting logic must be extremely high. Allowing both ways of forming groups in a single extended language would refer to the total lattice structure of the fibres $\mathcal{A}_X$, for an arbitrary set X. In general, for an arbitrary topological category, the structure of such lattices is extremely complex. For instance, in the case of topological spaces, see the paper[51]. ◀

3.4 Agents as Empirical Variables

In this section, we will discuss an alternative view of agents as *empirical variable maps* using topological categories, which, in some sense, unifies all the descriptions of extended modal logic with dependence atoms and group agents in the previous sections in a very nice way. Such a treatment is very much inspired by the paper[6], and in particular, we will show how the development there could be vastly generalised to the abstract framework for arbitrary topological categories.

Let $(\mathcal{A}, |-|)$ be a topological category. Recall in Section 2.3 and 2.4, we have discussed initial and final liftings of structured sources and sinks in topological categories. Recall that, a structured source S is a family of functions of the following type,

$$\mathsf{S} = \{f_i : X \to |A_i|\}_{i \in I},$$

where X is a set and A_i is an object in $\mathcal{A}$, for any $i \in I$. An initial lift of S is an object A in $\mathcal{A}_X$, which intuitively could be thought of as the universal object that makes every function $f : |A| \to |A_i|$ an $\mathcal{A}$-morphism, for any $i \in I$.

In particular, for any single structured source $f : X \to |B|$, we can perform the initial lift of f, and obtain an object A in the fibre $\mathcal{A}_X$. Trivially, every object in the fibre $\mathcal{A}_X$ can be thought of this way:

Lemma 3.7: For any topological category $\mathcal{A}$, every object in the fibre $\mathcal{A}_X$ for any set X is an initial lift of a single structured source

$$f : X \to |B|.$$

Proof For any A in $\mathcal{A}_X$, simply take the structured source

$$1_X : X \to |A| .$$

It is easy to verify that the initial lift of this structured source is A itself. ∎

In fact, we can further require such single structured source to be surjective or injective, since identities are trivially both surjective and injective. Hence, we can simply define a structured source based at X as an *empirical variable* with the set of states X, and Lemma 3.7 then shows that we can identify an agent with such an empirical variable (though this identification is not unique in any essential way). Furthermore, we will simply identify the empirical variable function as the agent name. Hence, in the future text of this section, an agent a (based on X) is the same as an empirical variable

$$a : X \to |B| ,$$

which is also identified with its initial lift. Also recall that, using the notation we have introduced in Section 2.4, the initial lift of a could also be written as $a^* B$; similarly, if we have structured sink $b : |B| \to Y$, then the final lift of it in $\mathcal{A}_Y$ is denoted as $b_! B$. For any function $f : X \to Y$, recall that forming final and initial lifts defines an adjunction between fibres,

$$f_! \dashv f^* .$$

More generally, for any indexed set Σ, an interpretation of the agents in the topological category $\mathcal{A}$ is a Σ-indexed tuple $(A_a)_{a\in\Sigma}$ in $\mathcal{A}^\Sigma$ over some set X; Lemma 3.7 then suggests that it can be equally described as a structured source on X indexed by Σ,

$$\{a : X \to |B_a|\}_{a\in\Sigma} .$$

A particular nice point about using empirical variables as description of agents is that, it is quite easy to specify the $\bigwedge$-combination of the group agent:

Proposition 3.14: Let $\{a : X \to |B_a|\}_{a\in\Sigma}$ be a Σ-indexed empirical variables with common source X, and let A_a be the initial lift $a^* B_a$ in $\mathcal{A}_X$ for each $a \in \Sigma$. Then we have the $\bigwedge$-combination $\bigwedge_{a\in\Sigma} A_a$ is given by the initial lift of the following single structured source,

$$\prod_{a\in\Sigma} a : X \to \left|\prod_{a\in A} B_a\right| .$$

Proof By Lemma 2.5, we know that in a topological category $\mathcal{A}$, if A_a is the initial lift of $a : X \to |B_a|$ for any $a \in \Sigma$, then $\bigwedge_{a\in\Sigma} A_a$ gives the initial lift of the following structured source

$$\{a : X \to |B_a|\}_{a\in\Sigma}.$$

Now by the proof of Theorem 2.2, we also know that, when Σ is a set, the initial lift of the above source could be realised as the initial lift of the following single structured source,

$$\prod_{a\in\Sigma} a : X \to \left|\prod_{a\in A} B_a\right|.$$

This shows that, the $\bigwedge$-combination of agents, in the description of empirical variables, is precisely given by the product of the individual empirical variables. ∎

This is exactly the LCD style of treating groups as a single agent, by taking the product of the codomain of each empirical variable, and performing the initial lift. However, our general description using topological categories shows that, the same approach could be adopted for other examples of topological categories, as the more familiar **Kr** or the more general **Nb**.

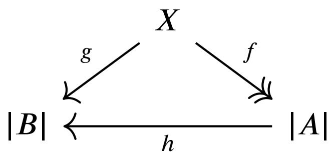
Furthermore, there are also equivalent description of the semantics of dependence atoms using empirical variables. To obtain a complete analogy with that in[6], we first define the notion of separated objects in a topological category:

Definition 3.6: Let $(\mathcal{A}, |-|)$ be a topological category. An object B in $\mathcal{A}$ is said to be *separated*, if for any set X, object A in $\mathcal{A}$, and any surjection $f : X \twoheadrightarrow |A|$ and any function $g : X \twoheadrightarrow |B|$, if $f^*A \leq g^*B$, then there exists a function $h : |A| \to |B|$ making the following diagramme commute,

$$\begin{array}{ccc} & X & \\ {\scriptstyle g}\swarrow & & \searrow{\scriptstyle f} \\ |B| & \xleftarrow[h]{} & |A| \end{array}$$

In fact, we can prove that if such a factorisation exists, then h itself must be an $\mathcal{A}$-morphism:

Lemma 3.8: Let $(\mathcal{A}, |-|)$ be a topological category. If we have a factorisation in **Set** as follows,



then h is an $\mathcal{A}$-morphism iff $f^*A \leq g^*B$.

Proof On the one hand, suppose h is an $\mathcal{A}$-morphism, then it follows that $A \leq h^*B$, which implies that

$$f^*A \leq f^*h^*B = (hf)^*B = g^*B.$$

This uses the functoriality of initial lifts (cf. Section 2.4).

On the other hand, suppose $f^*A \leq g^*B$. Use the axiom of choice, find any section $s : |A| \to X$ of f satisfying $f \circ s = 1_{|A|}$. Then we have

$$h = h \circ f \circ s = g \circ s.$$

Hence, it follows that

$$h^*B = s^*g^*B \geq s^*f^*A = (fs)^*A = A.$$

This then shows that h is indeed an $\mathcal{A}$-morphism. ∎

For concreteness, here we characterise separated objects in some example categories. First, it should be easy to see that an object in **Eqv** is separated iff it is a discrete equivalence relation. As a more general example, we first characterise separated objects in **Pre**:

Lemma 3.9: A preorder is separated in **Pre** iff it is a partial order.

Proof Suppose a preorder P is separated, with the underlying set $X = |P|$. Let Q be its partial order skeleton, i.e. the quotient $Q = P/\approx$ where the equivalence relation $\approx$ is given by

$$x \approx y \Leftrightarrow x \leq y \,\&\, y \leq x.$$

Then obviously, we have a surjection $X \twoheadrightarrow |Q|$, by sending each $x \in X$ to the equivalence class $[x]$ in Q. Moreover, the initial lifts of the two surjections $1_X : X \twoheadrightarrow |P|$ and $X \twoheadrightarrow |Q|$ are the same, viz. P. This suggests that there exists a function $|Q| \to |P|$, such that the following map compose to identity,

$$|P| \xrightarrow{/\approx} |Q| \longrightarrow |P|$$

Since $|Q|$ is a quotient of $|P|$, this suggest that $|Q|$ must be isomorphic to $|P|$, which means that P must already be a partial order from the start.

On the other hand, suppose P is a partial order, and suppose we have a function $g : X \to |P|$ and a surjection $f : X \twoheadrightarrow |Q|$ such that $f^*Q \leq g^*P$. In particular, this means that for any $x, y \in X$,

$$f(x) \leq f(y) \Rightarrow g(x) \leq g(y).$$

Now suppose we have a pair $x, y \in X$ that $f(x) = f(y)$. This then suggests that $g(x) \leq g(y)$ and $g(y) \leq g(x)$. Since P is a partial order, we must then have that $g(x) = g(y)$. This means that g respects the equivalence relation generated by f, thus there exists a function $h : |Q| \to |P|$ such that

$$g = h \circ f. \qquad \blacksquare$$

Once we have established Lemma 3.9, having identified the separated objects in **Pre**, we can use the vertical connection between **Top** and **Pre**, viz. an adjunction between them, to easily find the separated objects in **Top**. However, the relevant functors between them will only be introduced in Chapter 4. Here we prove the result in **Top** by hand one more time, but through the proof you can already identify its close connection with the case in **Pre**:

Lemma 3.10: A topological space is separated in **Top** iff it is T_0.

Proof Suppose T is a separated topological space with underlying set X. Recall from Example 2.2 (iv). there is an associated specialisation order $\preceq$ on T. We then quotient T by the equivalence relation generated by this preorder $\preceq$ as what we did in the proof of Lemma 3.9, and call the resulting quotient space endowed with the quotient topology S. Then obviously, there are surjections $1_X : X \twoheadrightarrow |T|$ and $X \twoheadrightarrow |S|$, which has the same initial lift on X, viz. T. This again suggests that the following composition is identity,

$$|T| \longrightarrow |S| \longrightarrow |T|$$

hence we must have S and T have the same underlying set, i.e. $\preceq$ is a partial order. This exactly means that T is a T_0 space.

On the other hand, suppose T is T_0. Suppose we have a function $g : X \to |T|$ and a surjection $f : X \twoheadrightarrow |S|$, such that $f^*S \leq g^*T$. Suppose for any $x, y \in X$ we have $f(x) = f(y)$. Now for any open neighbourhood U of $g(x)$ in T, by definition $g^{-1}(U)$ is open in g^*T. Since $f^*S \leq g^*T$, it follows that $g^{-1}(U)$ is also open in f^*S, hence there exists an open neighbourhood V of $f(x)$, such that $f^{-1}(V) = g^{-1}(U)$. However, since $f(x) = f(y)$, it follows that we also have $y \in f^{-1}(V) = g^{-1}(U)$, which means that U contains $g(y)$ as well. By symmetry, this suggest that any open neighbourhood of $g(x)$ is an open neighbourhood of $g(y)$, and vice versa. Since T is T_0, this suggest that $g(x)$ and $g(y)$ are equal, and hence g respects the equivalence relation generated by f, and there exists a factorisation as desired. ∎

Corollary 3.4: In **Eqv**, **Pre** and **Top**, every object A over a set X is an initial lift of some empirical variable $f : X \to |B|$, where B is separated.

Proof This can be implied from the proof of Lemma 3.8 and 3.9. ∎

Then, at least in the case of **Eqv**, **Pre** and **Top**, it is safe to assume that each empirical variable is surjective, with the codomain coming from the underlying set of a separated object. Assuming this condition, we then obtain the following generalisation of results in[6]:

Proposition 3.15: For any two empirical variables based at X

$$a : X \twoheadrightarrow |B_a|, \quad b : X \twoheadrightarrow |B_b|,$$

the following conditions are equivalent:

(i). Agent b globally depends on a, viz. $X \models K_a b$.

(ii). In $\mathcal{A}_X$, $a^*B_a \leq b^*B_b$.

(iii). The map $b : |a^*B_a| \to |B_b|$ is an $\mathcal{A}$-morphism.

And if B_b is furthermore separated, they are further equivalent to:

(iv). There exists an $\mathcal{A}$-morphism $g : |B_a| \to |B_b|$ such that $g \circ a = b$.

Proof (i) ⇔ (ii) by definition. (ii) ⇔ (iii) by the definition of initial lift. We now show (iv) ⇒ (iii): Since $g : |B_a| \to |B_b|$ is an $\mathcal{A}$-morphism, it follows that $B_a \leq g^*B_b$, hence we also have

$$a^*B_a \leq a^*g^*B_b = (ga)^*B_b = b^*B_b.$$

This uses the functoriality of initial lifts. (ii) ⇒ (iv) is also immediate now, when B_b is supposed to be separated. By the definition of separatedness, there exists a factorisation $g : |B_a| \to |B_b|$ such that $g \circ a = b$. By Lemma 3.8, this g itself must also be an $\mathcal{A}$-morphism. ∎

In principle, We can also develop a local version analogues to Proposition 3.15, which provides still more connection with LCD presented in[6]. However, to generalise the results there in the language of topological categories is not the primary goal of this thesis. We hope that the materials gathered here should have convinced the readers that, topological categories ought to be a very nice abstract framework to develop LCD-like modal systems in a much more general context.

Chapter 4 Vertical Connections within the Landscape

In the previous chapter, we are essentially studying topological categories equipped with a semantic functor. In different sections we have shown that, any such topological category will support the interpretation of basic modal language $\mathcal{L}$, or its various extended form $\mathcal{L}_\Sigma$, $\mathcal{L}_\Sigma^D$, $\mathcal{L}_{\Sigma_l}^D$ or $\mathcal{L}_{\Sigma_r}^D$. These interpretation are in particular related to the *internal structures* of a topological category, e.g. the available structure in the fibre and between the fibres by final or initial lifts, and the specific property of the semantic functors. And in each case, we have shown how the concrete examples we have in mind fits into our abstract and general description, and have shown most of them have very nice properties, in the sense that there are various dualities realised by the semantic functors.

In this chapter, we will take a bird's-eye view for the above individual examples, and turn to the study of Problem 1.2. In particular, we will show how the various examples of topological categories we have considered so far, which serves as different classes of models for different fragment of modal logics, connect to each other. How different types of vertical connections interacts with the interpretation of modal languages would also be a major theme.

But as before, to facilitate this process it is convenient to first develop the general and abstract framework of vertical connections, and only dive into concrete examples later on. Following the discussion in Chapter 1, this means that we need to study the category formed by all possible information structures that provide interpretation for modal logic. We hope that, through the development and discussion in Chapter 3, it is evident that the theory of topological categories, with the notion of semantic functors, would be a good unifying framework for the various modal systems we have in mind. Hence, we will provide a mathematical study of the categorical structure of $\mathfrak{Topc}$, the category of all topological categories itself. We will identify the category of those topological categories supporting modal reasoning as certain slice category $\mathfrak{Topc}/\mathbf{CABAO}^{\mathrm{op}}$ over $\mathbf{CABAO}^{\mathrm{op}}$, whose categorical structure should be able to derive immediately from that of $\mathfrak{Topc}$.

The aim of the remaining part of this chapter is then to apply this general investigation to the study of concrete examples we have in mind. More specifically, we aim to provide a detailed description of the following diagramme, which is a more refined description of the vertical aspect presented in Figure 1.1 in Chapter 1:

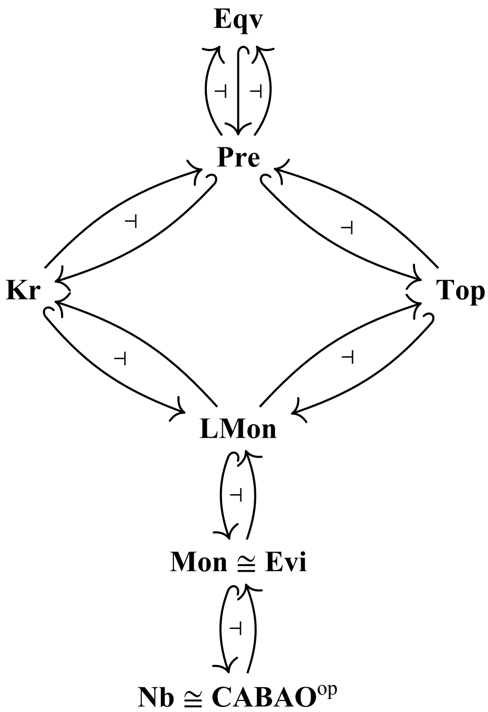


Figure 4.1 Vertical Connection within the Landscape

Let us first provide an overall explanation of what Figure 4.1 represents. First of all, we have the *skeleton* of this diagramme, which are those arrows with hooks. They all signify *fully faithful embeddings*, and they all commutes with the semantic functors we have constructed for these topological categories in Section 3.1. This makes the information structures at higher levels be able to be viewed as special cases within the ones at a lower level. As we will see later, this in particular implies that any semantic model in the higher level induces the *same* interpretation when viewing them as objects in the lower levels through such embeddings, for any formula *in a suitable language*. Generally speaking, any such full embedding will at least preserve the interpretation of the language $\mathcal{L}^D_\Sigma$, as we will show in Section 4.5. However, whether it will further preserve richer fragments of the previously discussed modal languages, such as $\mathcal{L}^D_{\Sigma_l}$ or $\mathcal{L}^D_{\Sigma_r}$, depends on the mathematical properties of the embedding.

The properties of such fully faithful embeddings are particularly represented by the existence of a left or right adjoint, sometimes even both. For our concrete examples, the left or right adjoints are indicated in Figure 4.1. These left or right adjoints typically will *not* commute with the semantic functors of the associated domain and codomain. Thus, usually they will not preserve the interpretation of the modalities. Nonetheless, they are all concrete functors, i.e. they preserve the underlying set. These concrete left or right adjoints identify models at higher levels as (co)reflexive subcategories of models at lower levels, with the (co)reflection telling us what the best approximate of an object in the ambient category is, using objects in the smaller subcategory. In Section 4.5, we will further show that the existence of such left or right adjoint implies richer fragments of modal languages.

The existence of such adjunctions suggests that there are very close interconnections between the different levels. And as we will see in the general framework of topological categories in Section 4.1, such adjunctions are abundant between topological categories. Though every hooked arrows in Figure 4.1 has a left or right adjoint, as indicated above, not every full embedding between topological categories have an adjoint. This could already be seen in the above diagramme, by composing two full embeddings. For example, we present the full embedding **Top** $\hookrightarrow$ **Mon** in Figure 4.1 as factorising another inter-level category **LMon**, which we will introduce in due course. **Top** $\hookrightarrow$ **LMon** has a left adjoint, while **LMon** $\hookrightarrow$ **Mon** has a right adjoint, which makes the embedding **Top** $\hookrightarrow$ **Mon** lacking direct left or right adjoints, but can only be factorised in this way. Other examples including the embedding **Pre** $\hookrightarrow$ **Mon**. In Section 4.2, we will investigate the general mathematical properties of full embeddings between topological categories, which provides a unified description of all such embeddings as certain factorisations with functors possessing left or right adjoints.

In fact, there are much more items that could fit into the above landscape. The category $\mathfrak{Topc}$, or the slice category $\mathfrak{Topc}/\mathbf{CABAO}^{\mathrm{op}}$ which contains those topological categories with a semantic functor, have infinitely many objects, and the above Figure 4.1 only represents a tiny part of the total picture. For instance, various domain theoretic models, like Scott domains, naturally sit above **Top**, which should appear in the up right corner. Though domains are *a priori* posets, we generally do not consider the inclusion of domains into the category **Pre**, since this inclusion is not full. In other words, the notion of morphisms in domain theory makes the category of domains connected to topological spaces much closer than to orders.

More generally, we can look at topological categories over *n*-fold products of

CABAO$^{\text{op}}$ itself, which generally allows us to combine different types of modalities in the same language. For instance, in Example 3.2 we have shown that a certain class of monotone neighbourhood frames supports a further doxastic modality, modelling the possibility of belief. In those cases, we use the product construction presented in Chapter 2 to derive new models out of the existing ones, which supports two modes of modal reasoning, and the interaction between the two. Another example is the plausibility models, which are sets equipped with both an equivalence relation and a preorder. Hence, in some sense, the above diagramme could be thought as basic ingredients, that could be combined in different ways that will result in new systems.

The organisation of this chapter is similar in spirit to the previous chapter. As mentioned, we will first provide a mathematical study of the structure of the category of topological categories itself in Section 4.1. We will then identify certain canonical scenarios that appear in Figure 4.1, covering most of the vertical connections we would like to consider, and study their properties with respect to the interpretation of modal language. This will be the content of the remaining sections. We will discuss in much more details into study the concrete arrows in Figure 4.1 in these sections.

4.1 $\mathfrak{Topc}$ as a 2-Category

To lay a proper mathematical background in the general study of vertical connections within the landscape of information we have motivated in Chapter 1, we must take a step further, to study the structure of the category $\mathfrak{Topc}$ of topological categories itself. We do this by first describe $\mathfrak{Conc}$, the category of concrete categories, as we have defined in earlier sections, and then identify $\mathfrak{Topc}$ as the subcategory of $\mathfrak{Conc}$ with objects as topological categories, and morphisms as concrete functors of certain special properties.

The first thing to notice is that, $\mathfrak{Conc}$ and $\mathfrak{Topc}$ are not only categories, but in fact *2-categories* in the sense of[41]. Or equivalently, they are enriched over **Cat** in the sense of[52]. Roughly, this means that for any two topological categories, $\mathcal{A}, \mathcal{B}$, the morphisms from $\mathcal{A}$ to $\mathcal{B}$ in $\mathfrak{Topc}$ is not only a set, but a *category* in itself. However, it is much easier than ordinary 2-categories, in that, according to our definition, $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$ will only be a preorder. To see this, we first define what 2-cells in $\mathfrak{Conc}$ we would like to consider:

Definition 4.1 (Concrete Natural Transformation): Let $F, G : \mathcal{A} \to \mathcal{B}$ be two concrete functors in $\mathfrak{Conc}$. A *concrete natural transformation* $\tau : F \to G$ is a natural transformation τ, such that for any $A \in \mathcal{A}$, $|\tau_A| = 1_{|A|}$.

Lemma 4.1: For any concrete functors $F, G : \mathcal{A} \to \mathcal{B}$, there at most exists a unique concrete natural transformation from F to G, and this happens exactly when $FA \leq GA$, for any A in $\mathcal{A}$.

Proof By defintion, concrete natural transformations has components over the identity functions, thus for any $A \in \mathcal{A}_X$, the components $\tau_A : FA \to GA$ must be in the fibre $\mathcal{B}_X$, which is unique since $\mathcal{B}_X$ is a preorder. Hence, we only need to show that when $FA \leq GA$ for any $A \in \mathcal{A}$, this indeed gives a natural transformation, i.e. for any $f : A \to B$ in $\mathcal{A}$, the following diagramme commutes,

$$\begin{array}{ccc} FA & \xrightarrow{\leq} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\leq]{} & GB \end{array}$$

It commutes by the fact that F, G are concrete functors, thus the two compositions has the same underlying function. Then by faithfulness of the forgetful functor, they must coincide as morphisms in $\mathcal{B}$. ∎

According to Lemma 4.1, we simply write $F \leq G$ when there exists a concrete natural transformation from F to G, or equivalently when $FA \leq GA$ for any A in $\mathcal{A}$. This fact implies our original claim that $\mathfrak{Conc}$ is a 2-category:

Corollary 4.1: $\mathfrak{Conc}$ is a 2-category enriched over preorders, such that for any two concrete categories $\mathcal{A}, \mathcal{B}$, $\mathfrak{Conc}(\mathcal{A}, \mathcal{B})$ is the set of functors from $\mathcal{A}$ to $\mathcal{B}$, preordered by concrete natural transformations.

Proof Straight forward verification. ∎

We would then expect that, the category of topological categories $\mathfrak{Topc}$, should be a 2-subcategory of $\mathfrak{Conc}$, but *not* a full one. This is because, for topological categories, we have much more structures than a mere construct, in that we can perform initial or final lifts for structured sources or sinks. The correct notion of morphisms between them should then respect these structures in a coherent way. To motivate and prepare for our definition for the morphisms in $\mathfrak{Topc}$, let's first prove the following result:

Proposition 4.1: Let $\mathcal{A}, \mathcal{B}$ be two topological categories, and let $F : \mathcal{A} \to \mathcal{B}$ be a concrete functor. Then the following conditions are equivalent:

1. F preserves final sinks;
2. F has a concrete right adjoint G;
3. F has a right adjoint in the 2-category $\mathfrak{Conc}$, i.e. a concrete functor $G : \mathcal{B} \to \mathcal{A}$ such that $1_{\mathcal{A}} \leq G \circ F$, and $F \circ G \leq 1_{\mathcal{B}}$.

Completely dually, the following conditions are also equivalent:

1. F preserves initial sources;
2. F has a concrete left adjoint G;
3. F has a left adjoint in the 2-category $\mathfrak{Conc}$, i.e. a concrete functor $G : \mathcal{B} \to \mathcal{A}$ such that $1_{\mathcal{A}} \leq F \circ G$, and $G \circ F \leq 1_{\mathcal{B}}$.

Proof Again, we only prove the case for F preserving final sinks and the existence of right adjoint. The other case follows from duality. (1) ⇒ (3): Suppose F preserves all final sources. In particular, by Proposition 2.2 it follows that, for any set X, fibre-wise $F_X : \mathcal{A}_X \to \mathcal{B}_X$ preserves arbitrary joins, and that F commutes with final lifts $f_!$, for any function f. Since both $\mathcal{A}_X$ and $\mathcal{B}_X$ are complete lattices, F_X preserves arbitrary join implies that it has a right adjoint $G_X : \mathcal{B}_X \to \mathcal{A}_X$. Notice that F commutes with $f_!$ for any function $f : X \to Y$ means the following diagramme commutes,

$$\begin{array}{ccc} \mathcal{A}_X & \xrightarrow{F_X} & \mathcal{B}_X \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ \mathcal{A}_Y & \xrightarrow[F_Y]{} & \mathcal{B}_Y \end{array}$$

which implies that all of their right adjoints also commute,

$$\begin{array}{ccc} \mathcal{A}_X & \xleftarrow{G_X} & \mathcal{B}_X \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ \mathcal{A}_Y & \xleftarrow[G_Y]{} & \mathcal{B}_Y \end{array}$$

which by Proposition 2.2 again, it follows that G is a concrete functor $G : \mathcal{B} \to \mathcal{A}$, which preserves arbitrary initial sources. From the construction it is immediate to see that $1_{\mathcal{A}} \leq G \circ F$ and $F \circ G \leq 1_{\mathcal{B}}$.

(3) ⇒ (2): According to our definition of 2-cells in $\mathfrak{Conc}$, concrete natural transformations are, *a priori*, natural transformations. Hence, any functor G satisfies $1_{\mathcal{A}} \leq G \circ F$ and $F \circ G \leq 1_{\mathcal{B}}$ will be a concrete right adjoint of F.

(2) ⇒ (1): Since F has a concrete right adjoint, then it follows that, fibre-wise, any $F_X : \mathcal{A}_X \to \mathcal{B}_X$ preserves arbitrary joins. Now for any $f : X \to Y$ and any $A \in \mathcal{A}_X$, by functoriality of F we already know that

$$f_! FA \leq F f_! A,$$

this is because $f : |A| \to |f_! A|$ is by definition an $\mathcal{A}$-morphism. Hence, we only need to show the reverse inequality holds. By the concrete adjunction $F \dashv G$, it follows that

$$F f_! A \leq f_! FA \Leftrightarrow f_! A \leq G f_! FA.$$

We also know that

$$G f_! FA \leq GF f_! A,$$

and that

$$f_! A \leq GF f_! A.$$

The first follows from the previous inequality between F and $f_!$, and the second follows from the adjunction $F \dashv G$. Thus, indeed we have F also commutes with $f_!$, for any $f : X \to Y$. This completes the proof that F preserves all final sinks. ∎

With Proposition 4.1 in hand, we may define the 2-category of topological categories $\mathfrak{Topc}$ as follows:

Definition 4.2 (The 2-Category of Topological Categories): The 2-category of topological categories $\mathfrak{Topc}$ has objects as topological categories. A morphism $F : \mathcal{A} \to \mathcal{B}$ between two topological categories is a pair of concrete adjoint functors

$$F_! : \mathcal{A} \rightleftarrows \mathcal{B} : F^*.$$

The 2-cells are inherited from $\mathfrak{Conc}$, i.e. we write $F \leq G$ for two morphisms in $\mathfrak{Topc}$ iff $F_! \leq G_!$.

By Proposition 4.1, a morphism $F : \mathcal{A} \to \mathcal{B}$ is equivalently a concrete functor from $\mathcal{A}$ to $\mathcal{B}$ that preserves all final sinks. This identifies with the $F_!$ part of the functor for a morphism according to Definition 4.2. Hence, if $F : \mathcal{A} \to \mathcal{B}$ is a morphism in $\mathfrak{Topc}$, we also use F to denote the left adjoint $F_!$, when no confusion would arise.

The 2-category structure in $\mathfrak{Topc}$ is then inherited from $\mathfrak{Conc}$, by considering those left adjoints, or functors preserving all final sinks. Now the identity functors obviously preserve all final sinks; for any functor $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, both of which preserve final sinks, their composition $G \circ F$ also preserves final sinks. Hence, $\mathfrak{Topc}$ is well-defined.

The most important observation is that $\mathfrak{Topc}$ is actually biequivalent with the functor category [**Set**, **SupL**], the category of all functors from **Set** to **SupL**. Notice that **SupL**, the category of suplattices, is enriched over itself, which implies that it is in fact a 2-category. This would then place the category [**Set**, **SupL**] of functors from **Set** to **SupL** as also a 2-category, with objects being functors, morphisms being natural transformations, and 2-morphisms being point-wise 2-morphisms inherited from the 2-categorical structure from **SupL**.

Strictly speaking, to say some category is enriched over **SupL**, we need to specify a certain symmetric monoidal closed structure on **SupL**, which is left out here. The tensor product on **SupL** is standard in the literature[53]. For us, this enrichment could be understood in elementary terms. A category C is enriched over **SupL** simply means that, for any objects X, Y in C, the set of morphisms $C(X, Y)$ from X to Y in C is a suplattice; and for any morphism $f : X \to Y$, both of the induced function

$$(-) \circ f : C(Y, Z) \to C(X, Z), \quad f \circ (-) : C(Z, X) \to C(Z, Y),$$

for any Z in C, are suplattice morphisms, i.e. they preserve joins in these suplattices of hom-sets. We first prove the equivalence of $\mathfrak{Topc}$ with [**Set**, **SupL**]:

Proposition 4.2: There is an equivalence of 2-categories of the following form,

$$\mathfrak{Topc} \cong [\mathbf{Set}, \mathbf{SupL}].$$

Proof From Theorem 2.4, a topological category $\mathcal{A}$ is equivalently a functor $\mathcal{A}_{(-)}$: **Set** $\to$ **SupL**. By Proposition 4.1, a morphism $F : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{Topc}$ is equivalently a concrete functor F from $\mathcal{A}$ to $\mathcal{B}$ that preserves all final sinks. By Proposition 2.2, such a functor is again equivalent to a natural transformation between the two functors $F : \mathcal{A}_{(-)} \to \mathcal{B}_{(-)}$. Finally by definition, for two morphisms $F, G : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{Topc}$, we have $F \leq G$ iff point-wise we have $FA \leq GA$ for any A in $\mathcal{A}$. This coincides with the 2-cells in [**Set**, **SupL**], since its 2-categorical structure is inherited from **SupL** by consider point-wise orders. This completes the proof that $\mathfrak{Topc}$ is equivalent to [**Set**, **SupL**] as a 2-category. ∎

Remark 4.1: Here we briefly indicate the historical development of topological categories behind Proposition 4.2. The bifibrational point of view of topological categories as we have discussed in Section 2.5 is already in the literature. The essential part of topological categories as certain fibrations is already left as an exercise in the manuscript[38], and further indicated more specifically in the relevant sections in[44]. Since every fibration admits Grothendieck construction, transferring a fibration over some category into an indexed category, the main content of Proposition 4.2 must already be, to some extent, implicit in the literature. However, to the limited knowledge of the author, no one has ever defined the category $\mathfrak{Topc}$ of topological categories as we do here in Definition 4.2, and the exact equivalence has never been stated explicitly. ◀

The equivalence presented in Proposition 4.2 is very convenient for us. In fact, Proposition 4.2 provides us another way of describing the 2-categorical structure of $\mathfrak{Topc}$, in that $\mathfrak{Topc}$ is actually enriched over **SupL**:

Corollary 4.2: $\mathfrak{Topc}$ is enriched over **SupL**.

Proof This follows immediately from the equivalence $\mathfrak{Topc} \cong [\mathbf{Set}, \mathbf{SupL}]$, and the fact that **SupL** is enriched over itself. ∎

We can also give a concrete description of the enrichment over **SupL**. Corollary 4.2 first means that for any topological categories $\mathcal{A}, \mathcal{B}$, the collection of morphisms from $\mathcal{A}$ to $\mathcal{B}$, denoted as $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$, is actually a suplattice, ordered by concrete natural transformation, or point-wise order in the fibre. The following lemma shows that joins in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$ can be calculated point-wise from the left adjoints, which in particular provides a concrete proof of $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$ being a suplattice:

Lemma 4.2: For any two topological categories $\mathcal{A}, \mathcal{B}$, the joins in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$ can be computed point-wise.

Proof For any family of morphisms $\{F_i\}_{i\in I}$ from $\mathcal{A}$ to $\mathcal{B}$ in $\mathfrak{Topc}$ considered as functors between them all preserving final sinks, consider its point-wise join $\bigvee_{i\in I} F_i$. Explicitly, for any A in $\mathcal{A}$ we have

$$\left(\bigvee_{i\in I} F_i\right)(A) = \bigvee_{i\in I} F_i A.$$

For any $f : X \to Y$ and any A, B in $\mathcal{A}_X$, $\mathcal{A}_Y$, respectively, if $f : |A| \to |B|$ is an $\mathcal{A}$

morphism, then by functoriality of each F_i it follows that, for any $i \in I$,

$$f_! F_i A \leq F_i B,$$

which implies that

$$f_! \bigvee_{i \in I} F_i A = \bigvee_{i \in I} f_! F_i A \leq \bigvee_{i \in I} F_i B..$$

This means that $\bigvee_{i \in I} F_i$ is also functorial.

We should also proves that $\bigvee_{i \in I} F_i$ preserves all final sinks, which, by Proposition 2.2, is equivalent to show that $\bigvee_{i \in I} F_i$ should preserve all joins in the fibre and commutes with final lifts of single structured sinks. Now obviously, it preserves joins fibre-wise by the fact that taking joins commutes each other: For any $\{A_j\}_{j \in J}$ in $\mathcal{A}_X$ for some set X, we have

$$\left(\bigvee_{i \in I} F_i\right)\left(\bigvee_{j \in J} A_j\right) = \bigvee_{i \in I} \bigvee_{j \in J} F_i A_j = \bigvee_{j \in J} \left(\bigvee_{i \in I} F_i\right) A_j.$$

The final equality uses the fact that $\bigvee_{j \in J}$ commutes with $\bigvee_{i \in I}$. Now for any $f : X \to Y$, by naturality of each F_i with $f_!$, for any $A \in \mathcal{A}_X$ we also have

$$f_! \bigvee_{i \in I} F_i A = \bigvee_{i \in I} f_! F_i A = \left(\bigvee_{i \in I} F_i\right) f_! A.$$

Hence, $\bigvee_{i \in I} F_i$ is a functor which preserves all final lifts, and hence consists of a morphism from $\mathcal{A}$ to $\mathcal{B}$ in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$. ∎

$\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$ being a suplattice for any topological categories $\mathcal{A}, \mathcal{B}$ in particular means that there is a bottom element in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$. For any $\mathcal{A}, \mathcal{B}$, we universally denote the bottom element in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$ as the following adjunction,

$$\bot : \mathcal{A} \rightleftarrows \mathcal{B} : \top.$$

As functors, $\bot$ (resp. $\top$) takes anything in the fibre $\mathcal{A}_X$ (resp. $\mathcal{B}_X$) to the bottom (resp. top) element in the fibre $\mathcal{B}_X$ (resp. $\mathcal{A}_X$). It is then easy to verify that we have the adjunction

$$\bot \dashv \top.$$

Notice that, our notation here matches the one we have defined in Section 3.1.4 for the constant modalities on any topological category. In Chapter 2, we have called $\bot, \top$ the discrete and codiscrete functor, respectively, and denote them as disc and codisc. There the terminology bears more geometrical or topological intuition; here we exploit the fact that $\mathfrak{Topc}$ is enriched over **SupL**.

The fact that $\mathfrak{Topc}$ is enriched over **SupL** furthermore means that for any F in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$ and any C in $\mathfrak{Topc}$, both of the following induced functions are morphisms in **SupL**,

$$F_* : \mathfrak{Topc}(C, \mathcal{A}) \to \mathfrak{Topc}(C, \mathcal{B}), \quad F^* : \mathfrak{Topc}(\mathcal{B}, C) \to \mathfrak{Topc}(\mathcal{A}, C).$$

Explicitly, F_* is given by post-composing with F, and F^* is given by pre-composing with F. Both of them preserve joins because joins are computed fibre-wise in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$, and that F preserves fibre-wise joins by Proposition 4.1 again.

Remark 4.2: In the categorical literature, a category enriched over **SupL** is called a *quantaloid*, which is a multi-object extension of the notion of a *quantale*[54]. It has many uses in theoretical computer science[55], automata theory and process semantics[56], many valued and fuzzy logic[57], topos theory[53], and many others. Corollary 4.2 then at least hints at, it might be possible to connect our study of topological categories in the context of quantaloids to other branches of logic and theoretic computer science, but this is beyond the topic of the present project. We leave this aspect for future work. ◀

4.2 Factorisation of Functors Between Topological Categories

As shown in Section 3.1, for all the examples we have in mind, bearing information structure and a system of modal reasoning, they all admits a canonical fully faithful embedding into the topological category $\mathbf{CABAO}^{\mathrm{op}}$. Hence in this subsection, we study the general mathematical property of such embeddings, with potential application into our description of the concrete arrows presented in Figure 4.1.

Let $i : \mathcal{A} \hookrightarrow \mathcal{B}$ be a full embedding between two topological categories. Notice that i cannot necessarily be identified with a morphism in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$ or $\mathfrak{Topc}(\mathcal{B}, \mathcal{A})$, in that it may not preserve joins or meets, or commutes with initial or final lifts of single structured sources.

Another way to look at such embeddings is to view $\mathcal{A}$ as a substructure of $\mathcal{B}$. We may identify $\mathcal{A}$ along i as a full subcategory of $\mathcal{B}$. Then by Theorem 2.5, we have a canonically determined concrete functor $R : \mathcal{B} \to \mathcal{A}$, which fixes elements in $\mathcal{A}$. Then

i can be identified with a morphism in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$ (resp. $\mathfrak{Topc}(\mathcal{B}, \mathcal{A})$) iff we have $i \dashv R$ (resp. $R \dashv i$). As we discussed above, we do not always have such adjunctions between the two functors. However, we may factorise R to obtain such adjunctions, as we will see below.

Before we prove our results, we first introduce some terminology:

Definition 4.3 (Injection and Surjection in $\mathfrak{Topc}$): For any morphism $F : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{Topc}$, we say it is *injective*, if it is injective on objects. Similarly, we say it is *surjective*, if it is surjective on objects.

Notice that an injective morphism $i : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{Topc}$ actually consists of an adjunction,

$$i : \mathcal{A} \rightleftarrows \mathcal{B} : R.$$

We first show that if i is injective, then it is actually a fully faithful embedding:

Lemma 4.3: Any injective morphism $i : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{Topc}$ as a concrete functor from $\mathcal{A}$ to $\mathcal{B}$ is a full embedding.

Proof Since i is concrete, it is already faithful. By definition, i is also injective on objects. Thus, we only need to show i is full. We use the fact that i is a concrete left adjoint, hence preserves fibre-wise joins and final lifts of single structured sinks. For any function $f : X \to Y$ and any A in $\mathcal{A}_X$, A' in $\mathcal{A}_Y$, suppose that $f : |iA| \to \left|iA'\right|$ is a $\mathcal{B}$-morphism, i.e. we have

$$f_! iA = i f_! A \leq iA'.$$

Hence, it follows that

$$i(f_! A \vee A') = i f_! A \vee iA' = iA'.$$

We also know that i is injective on objects, thus we have

$$f_! A \vee A' = A',$$

which means that $f_! A \leq A'$, or equivalently, $f : |A| \to \left|A'\right|$ is also an $\mathcal{A}$-morphism. Thus, i is also full, and $i : \mathcal{A} \to \mathcal{B}$ then consists of a full embedding between two topological categories. ∎

Remark 4.3: At this point, there is a possible notation clash for full concrete embeddings between two topological categories, and an actual injection in the category $\mathfrak{Topc}$. The difference is that, for the former, it does not necessarily possess a right adjoint; it could also possess a left adjoint, which will make it represented as a *quotient* in $\mathfrak{Topc}$, as we will see below. Hence, through out this paper, we will always use $\hookrightarrow$ for full concrete embeddings, while reserve $\rightarrowtail$ for proper injections in $\mathfrak{Topc}$. *Caveat Lector!* ◀

Similarly, we can show that any surjective morphism $q : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{Topc}$ has a fully faithful, injective on objects, right adjoint:

Lemma 4.4: If $q : \mathcal{A} \to \mathcal{B}$ is surjective morphism in $\mathfrak{Topc}$, then considered as a functor from $\mathcal{A}$ to $\mathcal{B}$, its right adjoint R is a fully faithful embedding.

Proof If q is surjective, by the fact that it is a left adjoint, fibre-wise it consists of quotients of suplattices,

$$q_X : \mathcal{A}_X \twoheadrightarrow \mathcal{B}_X.$$

This in particular means that the right adjoints are fibre-wise embeddings of suplattices,

$$R_X : \mathcal{B}_X \hookrightarrow \mathcal{A}_X.$$

Hence, R must then be fully faithful and injective on objects. ∎

Corollary 4.3: Every morphism in $\mathfrak{Topc}$ can be factorised in an essentially unique way as a surjection followed by an injection.

Proof Lemma 4.3 and Lemma 4.4 have shown us that, the injections and surjections in $\mathfrak{Topc}$ as Definition 4.3 specifies are simply fibre-wise injections and surjections between suplattices. Then by Proposition 4.2, since $\mathfrak{Topc}$ is a functor category [**Set**, **SupL**], the image factorisation of $\mathfrak{Topc}$ are computed fibre-wise in **SupL**. **SupL** is algebraic over **Set**, thus it has image factorisation of surjection followed by injections. $\mathfrak{Topc}$ then inherits this structure. ∎

Remark 4.4: In fact, simply by abstract category theory, the equivalence $\mathfrak{Topc} \cong$ [**Set**, **SupL**] actually means that $\mathfrak{Topc}$ inherits most of the nice properties from the category **SupL** of suplattices, including completeness and cocompleteness — as we will show in more detail later — and here the point-wise image factorisation from **SupL**. In fact, a functor category [**Set**, **SupL**] is, in some sense, as similar to **SupL** as it can be, in terms of categorical properties. For example, **SupL** is furthermore an *exact category*, or in other words an *effective regular category*, which means we not only have the image factorisation in **SupL**, but they behave in a particularly nice way and that quotients in **SupL** could be presented as congruence. The equivalence presented in Proposition 4.2 makes $\mathfrak{Topc}$ also an exact category. A detailed discussion of all the categorical properties of $\mathfrak{Topc}$ inherited from **SupL** is beyond the scope of this thesis, but hopefully, the readers would be convinced at this point that, $\mathfrak{Topc}$ has very nice categorical properties because of this equivalence. ◀

However, we not only would want to factorise the morphisms within $\mathfrak{Topc}$. As we will see, many of the concrete embeddings between the category of semantics we consider about will have either a left or a right adjoint, making it possible to describe them as either direct morphisms, or right adjoints of morphisms, in $\mathfrak{Topc}$. But we may also arrive at certain examples that lack such adjoints. We end this subsection by showing how we can describe general full concrete embeddings between topological categories, not necessarily possessing a left or right adjoint, as certain structures in $\mathfrak{Topc}$. The main result is the following theorem:

Theorem 4.1: If $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, and both $\mathcal{A}$ and $\mathcal{B}$ are topological categories, then $\mathcal{A}$ is a *subquotient* of $\mathcal{B}$ in $\mathfrak{Topc}$, i.e. there exists another topological category $\mathcal{C}$, such that we have a diagramme of the following form in $\mathfrak{Topc}$,

$$\begin{array}{ccc} \mathcal{C} & \overset{q}{\twoheadrightarrow} & \mathcal{A} \\ {\scriptstyle i}\downarrow & & \\ \mathcal{B} & & \end{array}$$

and that the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ can be realised as the right adjoint of q, followed by i,

$$i_! \circ q^* : \mathcal{A} \hookrightarrow \mathcal{B}.$$

Proof We identify $\mathcal{A}$ as a full subcategory of $\mathcal{B}$. Notice that though $\mathcal{A}_X$ is a subset of $\mathcal{B}_X$ for any X, the joins in $\mathcal{A}_X$ in general are not computed as joins in $\mathcal{B}_X$, otherwise the inclusion $\mathcal{A} \hookrightarrow \mathcal{B}$ might already be a morphism in $\mathfrak{Topc}$. Hence, we will use superscripts over $\bigvee$ to indicate where the joins are taken. This equally applies to final lifts of single structured sinks, thus we will also use superscripts to distinguish which category we perform the final lifts.

Let $\mathcal{C}$ be another full subcategory of $\mathcal{B}$, spanned by elements in $\mathcal{B}$ that could be written as final lifts *in* $\mathcal{B}$ of structured sinks in $\mathcal{A}$, i.e. those elements of the form

$$\bigvee_{i\in I}^{\mathcal{B}} (f_i)_!^{\mathcal{B}} A_i \in \mathcal{C}_X,$$

where $\{f_i : |A_i| \to X\}_{i\in I}$ is a structured sink that for each i we have A_i in $\mathcal{A}$. It is easy to see that $\mathcal{C}$ is also a topological category. The fibre-wise joins in $\mathcal{C}_X$ for any set X can be computed the same as in $\mathcal{B}_X$. This also simply corresponds to taking the union of a family of structured sinks. The final lifts of single structured sinks in $\mathcal{C}$ are also computed the same as in $\mathcal{B}$: For any $g : X \to Y$, we have

$$g_!^{\mathcal{B}} \bigvee_{i\in I}^{\mathcal{B}} (f_i)_!^{\mathcal{B}} A_i = \bigvee_{i\in I}^{\mathcal{B}} (g \circ f_i)_!^{\mathcal{B}} A_i.$$

Now for any B in $\mathcal{B}_Y$, by the universal property of final lifts, $g : \left|\bigvee_{i\in I}^{\mathcal{B}} (f_i)_!^{\mathcal{B}} A_i\right| \to |B|$ is a $\mathcal{B}$-morphism iff for any $i \in I$, $g \circ f_i : |A_i| \to |B|$ is a $\mathcal{B}$-morphism, iff the following holds,

$$\bigvee_{i\in I}^{\mathcal{B}} (g \circ f_i)_!^{\mathcal{B}} A_i \leq B.$$

Hence, indeed $\bigvee_{i\in I}^{\mathcal{B}} (g \circ f_i)_!^{\mathcal{B}} A_i$, or equivalently, $g_! \bigvee_{i\in I}^{\mathcal{B}} (f_i)_!^{\mathcal{B}} A_i$, is the final lift of $\bigvee_{i\in I}^{\mathcal{B}} (f_i)_!^{\mathcal{B}} A_i$ along g. Such a description not only proves that $\mathcal{C}$ is a topological category, it actually shows that the inclusion $\mathcal{C} \hookrightarrow \mathcal{B}$ is finally closed, thus consists of an injection in $\mathfrak{Topc}$.

Thus, we concludes the proof by providing a surjection q in $\mathfrak{Topc}$ from $\mathcal{C}$ to $\mathcal{A}$. For any C in $\mathcal{C}_X$, we let q_X map it to the following object in $\mathcal{A}_X$,

$$C \mapsto \bigvee^{\mathcal{A}} \{\, A \in \mathcal{A}_X \mid A \leq C \,\}.$$

This construction actually can be viewed as an interior operator on $\mathcal{C}_X$, because $\mathcal{A}_X$ also lives in $\mathcal{C}_X$. For any $A \in \mathcal{A}_X$, we obviously have $(1_X)_!^{\mathcal{B}} A = A$, thus $A \in \mathcal{C}_X$. Furthermore, the above mapping is obviously decreasing and idempotent, and obviously $\mathcal{A}_X$ is the set of its fixed-points. Then it follows from the theory of suplattices we have a well-defined surjection in **SupL**,

$$q_X : \mathcal{C}_X \twoheadrightarrow \mathcal{A}_X,$$

and its right adjoint is the inclusion $\mathcal{A}_X \hookrightarrow \mathcal{C}_X$. We show this defines a surjection $\mathcal{C} \twoheadrightarrow \mathcal{A}$

in $\mathfrak{Topc}$ by showing that $\mathcal{A}$ is initially closed in $\mathcal{C}$.

To this end, suppose we have an $\mathcal{A}$-structured source $\{f_i : X \to |A_i|\}_{i \in I}$, and let A be the initial lift of it in $\mathcal{A}$ over X. We need to show this is also an initial lift of this structured source in $\mathcal{C}$. Given any $g : |C| \to |A|$ where $|C| = Y$, suppose $f_i \circ g : |C| \to |A_i|$ are all $\mathcal{C}$-morphisms, hence $\mathcal{B}$-morphisms. Suppose now C is the final lift of the structured sink $\{g_r : |A_r| \to Y\}$ in $\mathcal{B}$. By definition, $g : |C| \to |A|$ is a $\mathcal{C}$-morphism, equivalently a $\mathcal{B}$-morphism, iff each $g_r \circ g : |A_r| \to |A|$ is a $\mathcal{B}$-morphism, equivalently an $\mathcal{A}$-morphism, iff each $g_r \circ g \circ f_i : |A_r| \to |C| \to |A_i|$ is an $\mathcal{A}$-morphism. Now we know that each $g_r : |A_r| \to |C|$ and $g \circ f_i : |C| \to |A_i|$ are $\mathcal{B}$-morphisms, hence the compositions are also $\mathcal{B}$-morphisms, hence $\mathcal{A}$-morphisms since $\mathcal{A}$ is a full subcategory. It then follows that A is indeed also the initial lift of the structured source in $\mathcal{C}$.

Finally, by definition of the injection $i : \mathcal{C} \hookrightarrow \mathcal{B}$ and the quotient $q : \mathcal{C} \twoheadrightarrow \mathcal{A}$, it is easy to see that the right adjoint q^*, which is the full embedding of $\mathcal{A}$ into $\mathcal{C}$, composed with the left adjoint $i_!$ embedding $\mathcal{C}$ into $\mathcal{B}$, is simply the embedding of $\mathcal{A}$ into $\mathcal{B}$. This completes the proof. ∎

The content of Theorem 4.1 is already contained inherently in[38], though it does not introduce the 2-category $\mathfrak{Topc}$ of topological categories as we do here, and hence does not view the decomposition as a subquotient.

Theorem 4.1 in particular provides a unifying description of all the canonical semantic functors we have constructed in Section 3.1, and certain other functors we have considered in Example 2.2, since all of them are full embeddings of some topological category of semantics of modal logic into our constructed topological category $\mathbf{CABAO}^{\mathrm{op}} \cong \mathbf{Nb}$, or other topological category of semantics. Now as indicated in Figure 4.1, many of them have direct left or right adjoints. The embedding of **Mon** into $\mathbf{CABAO}^{\mathrm{op}}$ even has both a left or right adjoint, making it simultaneously inject into $\mathbf{CABAO}^{\mathrm{op}}$ and also being a quotient. We will describe these concrete examples in Section 4.5 in much more detail.

However, as we've already mentioned at the start of this chapter, not all embeddings possess such an adjoint. One example is the full embedding $\mathbf{Top} \hookrightarrow \mathbf{Mon}$, which we have described in Example 2.2 (v). We will use it as a case study, showing that the above factorisation could be useful for our general theory of interconnections between different information levels. We provide a detailed description of it below:

Example 4.1 (Factorisation of the Full Embedding **Top** $\hookrightarrow$ **Mon**): We consider an intermediate topological category **LMon**, of *left exact and monotone* neighbourhood frames, i.e. those monotone neighbourhood frames (X, E) that is further closed under taking finite intersections: For any $x \in X$, we have xEX; and for any $S, T \subseteq X$, xES and xET implies that $xE(S \cap T)$. We will simply call such neighbourhood frames as *LM-neighbourhood frames*. The reason for this name is that, along the evident full embedding **LMon** $\hookrightarrow$ **Nb** $\cong$ **CABAO**$^{\mathrm{op}}$, the induced operators in the image of **LMon** are precisely those preserving finite intersections, or in terms of modal logic, those modalities satisfies Necessitation and the **K**-axiom.

There is an obvious full embedding **LMon** $\hookrightarrow$ **Mon**. We first show this full embedding preserves all final sinks, hence consists of an injection in $\mathfrak{Topc}$ from **LMon** to **Mon**. For any family of LM-neighbourhood relations $\{E_i\}_{i\in I}$ over X, it is easy to see that their intersection $\bigcap_{i\in I} E_i$ is again an LM-neighbourhood frame. For any function $f : X \to Y$ and any LM-neighbourhood relation E over X, recall from Example 2.6 that its final lift $f_! E$ on Y in **Mon** is given as follows: For any $y \in Y$ and $S \subseteq Y$,

$$yf_!ES \Leftrightarrow \forall x \in f^{-1}(y)[xEf^{-1}(S)].$$

Then obviously, $yf_!EY$ for any $y \in Y$, because $f^{-1}Y = X$ and, by definition, for any $x \in X$ we have xEX. Moreover, if $yf_!ES$ and $yf_!ET$, then for any $x \in f^{-1}(y)$, $xEf^{-1}(S)$ and $xEf^{-1}(T)$, which means that $xEf^{-1}(S \cap T)$, because $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, and E is left exact. This implies that **LMon** is finally closed in **Mon**, and thus we have an injection in $\mathfrak{Topc}$,

$$\mathbf{LMon} \rightarrowtail \mathbf{Mon}.$$

We now construct a surjection **LMon** $\twoheadrightarrow$ **Top** in $\mathfrak{Topc}$. Notice that the concrete embedding **Top** $\to$ **Mon** constructed in Example 2.2 (v). restricts to one **Top** $\to$ **LMon**, since open sets of topological spaces are closed under finite intersections. We show that this embedding has a concrete left adjoint. Given any LM-neighbourhood frame (X, E), we consider a topology on X, given by the following family of subsets,

$$\tau_E := \{\, S \subseteq X \mid S \subseteq E^{-1}(S) \,\}.$$

In other words, $S \in \tau_E$ iff for any $x \in S$, xES. It is easy to verify that $\varnothing, X \in \tau_E$.

This family τ_E is also closed under finite intersections. Suppose $S, T \in \tau_E$, then for any $x \in S \cap T$, $x \in S$ implies xES, and $x \in T$ implies xET. Since E is left exact, it follows that $xES \cap T$, hence $S \cap T \in \tau_E$. Finally, we show τ_E is closed under arbitrary union. Suppose we have a family $\{S_i\}_{i \in I}$ of subsets in τ_E. Then for any $x \in \bigcup_{i \in I} S_i$, by definition for some $i \in I$ we have $x \in S_i$. It then follows that xES_i, and by monotonicity, $xE \bigcup_{i \in I} S_i$. This meas that $\bigcup_{i \in I} S_i \in \tau_E$, and thus τ_E is a well-defined topology on X.

We show that this construction from an LM-neighbourhood frame on X to a topology on X defines a concrete left adjoint to the association of a topology τ on X to its associated neighbourhood relation N_τ as defined in Example 2.2 (v). To this end, we only need to show that for any function $f : X \to Y$, any LM-neighbourhood frame E on X and any topology τ on Y, $f : (X, \tau_E) \to (Y, \tau)$ is continuous iff $f : (X, E) \to (Y, N_\tau)$ is a morphism in **LMon**.

Suppose f is continuous between the two topological spaces. For any $x \in X$ and $S \subseteq Y$, if $fxE_\tau S$, then by definition there exists an open set U that $fx \in U \subseteq S$. It follows that $x \in f^{-1}(U)$ in X, and $f^{-1}(U)$ is open. By definition, $x \in f^{-1}(U)$ implies that $xEf^{-1}(U)$, which means that f is indeed a morphism between the two neighbourhood frames.

On the other hand, suppose that f is a morphism between the two neighbourhood frames. Then for any $x \in X$ and any open neighbourhood U in Y of fx, we have $fxN_\tau U$, which implies that $xEf^{-1}U$. This means that $x \in f^{-1}(U)$ implies that $xEf^{-1}(U)$, hence $f^{-1}(U)$ is an open set in τ_E. This in particular means that the function f is continuous locally at x, for any $x \in X$, or equivalently f is continuous.

The above completes the proof that the embedding **Top** $\hookrightarrow$ **LMon** has a concrete left adjoint, hence defines a surjection in $\mathfrak{Topc}$,

$$\mathbf{LMon} \twoheadrightarrow \mathbf{Top}.$$

Then, we have decomposed the concrete full embedding **Top** $\hookrightarrow$ **Mon** into the following subquotient of **Mon** in $\mathfrak{Topc}$,

$$\begin{array}{ccc} \mathbf{LMon} & \twoheadrightarrow & \mathbf{Top} \\ \big\downarrow & & \\ \mathbf{Mon} & & \end{array}$$

This completes the description promised before. ◀

We in fact have more examples on this line. For example, if we compose the embedding **Pre** $\hookrightarrow$ **Kr** and **Kr** $\hookrightarrow$ **Mon**, the embedding **Pre** $\hookrightarrow$ **Mon** would also lack a direct left or right adjoint. In this case, the factorisation is presented by **Kr** itself, resulting in the following diagramme,

$$\begin{array}{ccc} \mathbf{Kr} & \twoheadrightarrow & \mathbf{Pre} \\ \downarrow & & \\ \mathbf{Mon} & & \end{array}$$

Also notice that, by the description of **LMon** in Example 4.1, the embedding **Kr** $\hookrightarrow$ **Mon** obviously factors through **LMon**, which actually consists of successive injections in $\mathfrak{Topc}$ as follows,

$$\mathbf{Kr} \rightarrowtail \mathbf{LMon} \rightarrowtail \mathbf{Mon}$$

This and other examples will be treated in much more detail in Section 4.4 and Section 4.5.

Remark 4.5: We can say a little bit more about the process of finding this intermediate topological category **LMon**, given the full embedding **Top** $\hookrightarrow$ **LMon**. Notice that, as we've shown in Section 3.1.2, **Top** is concretely equivalent to the full subcategory of $\mathbf{CABAO}^{\mathrm{op}}$, consisting of those left exact, decreasing, and idempotent operators. To find the subquotient description of the embedding **Top** $\hookrightarrow$ **Mon** according to Theorem 4.1, we need to consider which of these properties are closed under final lifts, i.e. closed under taking joins in **Mon** — hence intersections in **Mon** — and taking final lifts of single structured sources. Being decreasing of monotone neighbourhood frame means that xES implies $x \in S$. It is easy to see that, though decreasing monotone frames are closed under taking intersections, it is not closed under taking final lifts of single structured sources. Idempotency in terms of neighbourhood frames means that xES implies $xE\{\, y \mid yES \,\}$. It is not closed under neither operations. Thus, the natural candidate is to consider **LMon**, the category of left exact monotone neighbourhood frames. ◀

Finally, we may present the concrete adjunctions appearing in Figure 4.1 as actual arrows in $\mathfrak{Topc}$. Notice that the existence of left or right adjoint of a full embedding will make it either a surjection or an injection in $\mathfrak{Topc}$, which will result in the following picture:

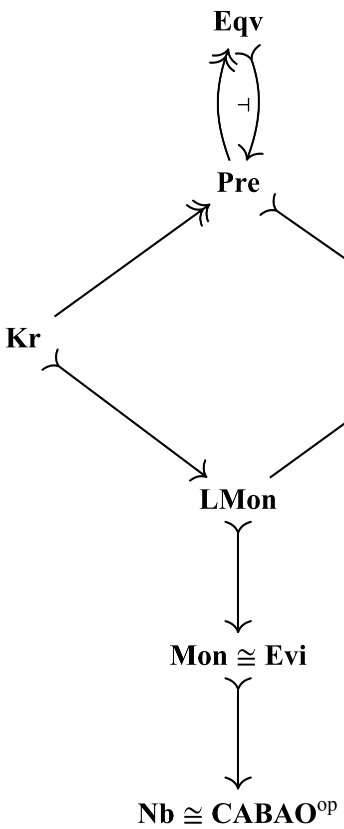


Figure 4.2 Vertical Connection as Morphisms in $\mathfrak{Topc}$

Notice that the adjunction now between **Eqv** and **Pre** are in the two category $\mathfrak{Topc}$, not in the usual 2-category of all categories. This is due to the fact that the embedding **Eqv** $\hookrightarrow$ **Pre** has both a left and right adjoint. The middle part of this diagramme forms a square in $\mathfrak{Topc}$, with two opposite sides being injections and surjections, respectively. We will show in later sections that it actually commutes in $\mathfrak{Topc}$. The tail and heads of the arrows in the above Figure 4.2 then also contains the information of the existence of either a left or right adjoint. As we've mentioned, these will be discussed in more detail in Section 4.5.

4.3 Limits and Colimits in $\mathfrak{Topc}$

This section proceeds to study the (2-)categorical properties of $\mathfrak{Topc}$ itself, showing various constructions of (2-)limits and (2-)colimits exists in $\mathfrak{Topc}$. As we will show below for the application of modal logic, such nice categorical properties of $\mathfrak{Topc}$ allow us to construct new categories of semantics of modal logic with desired properties.

4.3.1 Biproducts

In Section 2.2, we have shown that arbitrary products of topological categories in $\mathfrak{Conc}$ is again a topological category. In this section, we show that this structure in facts realises to *biproducts* in $\mathfrak{Topc}$, i.e. they are simultaneously products and coproducts. This is closely related to how we have specified the morphisms in Definition 4.2, and the fact that $\mathfrak{Topc}$ is enriched over **SupL**, as we have shown in Corollary 4.2. We will also investigate how to construct pullbacks, and more generally comma objects, in $\mathfrak{Topc}$, which has the potential to construct further semantics for modal logic via categorical structures.

We first show that biproducts exists in $\mathfrak{Topc}$. The following lemma gives explicit construction of such biproducts in $\mathfrak{Topc}$:

Lemma 4.5: $\mathfrak{Topc}$ has arbitrary biproducts, and the biproducts in $\mathfrak{Topc}$ is given by the products of concrete categories described in Section 2.2.

Proof From Theorem 2.2, we know that for any family of topological categories $\{\mathcal{A}_i\}_{i\in I}$, their products in $\mathfrak{Conc}$ is again a topological category, which we now denoted as $\bigoplus_{i\in I}\mathcal{A}_i$. We show it is the biproduct of this family.

We already know that it is a product of $\{\mathcal{A}_i\}_{i\in I}$ in $\mathfrak{Conc}$. To show it is the product of them in $\mathfrak{Topc}$, we only need to show all the projection maps have right adjoints. Recall that the projection map

$$\pi_i : \bigoplus_{i\in I}\mathcal{A}_i \to \mathcal{A}_i$$

maps any tuple $(A_i)_{i\in I}$ over the set X into A_i in $\mathcal{A}_X$. It is easy to see that the right adjoint can be defined by the following functor

$$\pi_i^* = \langle \top, \cdots, 1_{\mathcal{A}_i}, \cdots, \top\rangle : \mathcal{A}_i \to \bigoplus_{i\in I}\mathcal{A}_i,$$

where its i-th tuple is given by the identity functor on $\mathcal{A}_i$, and all the others are given by the top functor from $\mathcal{A}_i$ to $\mathcal{A}_j$, where $i \neq j$. It is easy to verify that it indeed gives the right adjoint of the i-th projection. This means that we have a well-defined projection map in $\mathfrak{Topc}$,

$$\pi_i : \bigoplus_{i\in I}\mathcal{A}_i \to \mathcal{A}_i.$$

We also construct the coprojection maps into $\bigoplus_{i\in I}\mathcal{A}_i$ explicitly. For any $\mathcal{A}_i$, the following is a well-defined functor

$$p_i = \langle \bot, \cdots, 1_{\mathcal{A}_i}, \cdots, \bot \rangle : \mathcal{A}_i \to \bigoplus_{i \in I} \mathcal{A}_i,$$

whose right adjoint is simply given by the projection

$$p_i^* = \pi_i : \bigoplus_{i \in I} \mathcal{A}_i \to \mathcal{A}_i.$$

Hence, we have also given a well-defined map in $\mathfrak{Topc}$,

$$p_i : \mathcal{A}_i \to \bigoplus_{i \in I} \mathcal{A}_i.$$

We left the readers to see that the structure maps π_i, p_i makes $\bigoplus_{i \in I} \mathcal{A}_i$ both the product and coproduct of the family $\{\mathcal{A}_i\}_{i \in I}$ in $\mathfrak{Topc}$. ∎

The description of biproducts in particular means that we have the following natural equivalence for any family of topological categories,

$$\mathfrak{Topc}\left(\bigoplus_{i \in I} \mathcal{A}_i, \bigoplus_{j \in J} \mathcal{B}_j\right) \cong \bigoplus_{i \in I, j \in J} \mathfrak{Topc}(\mathcal{A}_i, \mathcal{B}_j).$$

This simply follows from the universal property of biproducts. It means that, a morphism in $\mathfrak{Topc}$ from $\bigoplus_{i \in I} \mathcal{A}_i$ to $\bigoplus_{j \in J} B_j$ is given by an $I \times J$ matrix, with the i, j-entry provided by a single morphism from $\mathcal{A}_i$ to $\mathcal{B}_j$. This in fact generally holds for any category with biproducts.

Since any suplattice is a commutative monoid with a base change functor **SupL** $\to$ **CMon** when viewing binary join as the multiplication, thus $\mathfrak{Topc}$ is in particular also enriched over **CMon**, the category of commutative monoids. It is well-known that in any such category, finitary products coincide with finitary coproducts, if they exist. However, in some sense suplattices are certain *infinite* commutative monoids, in that arbitrary joins, rather than finite ones, exist in a suplattice. This facts directly contributes to the above theorem of the equivalence between arbitrary products and coproducts, and that $\mathfrak{Topc}$ has arbitrary biproducts.

In particular, this makes **Set** the *zero object* in $\mathfrak{Topc}$, i.e. it is both the terminal and the initial object. For any topological category $\mathcal{A}$, we have a unique morphism in $\mathfrak{Topc}$ mapping into **Set**, given by the following adjunction,

$$|-| : \mathcal{A} \rightleftarrows \mathbf{Set} : \top.$$

And similarly, we also have a unique morphism from **Set** to $\mathcal{A}$ in $\mathfrak{Topc}$, given by the following adjunction,

$$\bot : \mathbf{Set} \rightleftarrows \mathcal{A} : |-| .$$

These exploits the discrete and codiscrete structure present for every topological category.

We've already seen in Chapter 3 that the biproduct structure in $\mathfrak{Topc}$ is essential to describe systems of multi-agent modal logic, or to combine different interpretations of modalities, supporting an *n*-fold modal language.

4.3.2 Equalisers and Coequalisers

Since we have already shown that $\mathfrak{Topc}$ has all biproducts, which covers both arbitrary products and coporducts, to see $\mathfrak{Topc}$ is complete and cocomplete, we only need to provide the description of equalisers and coequalisers in $\mathfrak{Topc}$. As we've mentioned when showing the equivalence between $\mathfrak{Topc}$ and [**Set**, **SupL**], $\mathfrak{Topc}$ must be both complete and cocomplete because **SupL** is.

Proposition 4.3: $\mathfrak{Topc}$ has all equalisers.

Proof Given any two morphisms $F, G : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{Topc}$, we construct their equaliser $E : \mathcal{C} \rightarrowtail \mathcal{A}$ in $\mathfrak{Topc}$. For any set X, we define the fibre $\mathcal{C}_X$ as follows,

$$\mathcal{C}_X := \{ A \in \mathcal{A}_X \mid FA = GA \}.$$

Notice that since F, G are morphisms in $\mathfrak{Topc}(\mathcal{A}, \mathcal{B})$, they preserves arbitrary joins. This in particular means that $\mathcal{C}_X$ is closed under taking joins in $\mathcal{A}_X$: For any family $\{A_i\}_{i \in I}$ in $\mathcal{C}_X$, we have

$$F \bigvee_{i \in I} A_i = \bigvee_{i \in I} FA_i = \bigvee_{i \in I} GA_i = G \bigvee_{i \in I} A_i.$$

Since F, G also commutes with $f_!$, it follows that $\mathcal{C}$ is also closed under taking $f_!$: For any function $f : X \to Y$ and any $C \in \mathcal{C}_X$,

$$Ff_!C = f_!FC = f_!GC = Gf_!C.$$

Hence, $\mathcal{C}$ is finally closed in $\mathcal{A}$, hence there exists an injection in $\mathfrak{Topc}$,

$$E : \mathcal{C} \rightarrowtail \mathcal{A},$$

where E is simply the inclusion of $\mathcal{C}$ into $\mathcal{A}$. By construction, $F \circ E = G \circ E$. We leave the readers to check that E in fact defines the equaliser of F, G in $\mathfrak{Topc}$. ∎

For any pair $F, G : \mathcal{A} \to \mathcal{B}$ of morphisms in $\mathfrak{Topc}$, their coequaliser $E : \mathcal{B} \twoheadrightarrow \mathcal{C}$ can also be described in a similar fashion. For any set X, fibre-wise $\mathcal{C}$ can be defined as follows,

$$\mathcal{C}_X := \{ B \in \mathcal{B}_X \mid F^* B = G^* B \}.$$

Such construction in particular means that $\mathcal{C}$ is initially closed in $\mathcal{B}$, hence there is a surjection in $\mathfrak{Topc}$ from $\mathcal{B}$ to $\mathcal{C}$,

$$E : \mathcal{B} \twoheadrightarrow \mathcal{C},$$

whose right adjoint is the inclusion $\mathcal{C}$ into $\mathcal{B}$ as defined above. By construction, it is also easy to see that E is indeed the coequaliser of F, G in $\mathfrak{Topc}$.

Corollary 4.4: $\mathfrak{Topc}$ is complete and cocomplete as a 1-category.

Proof We know that $\mathfrak{Topc}$ has all biproducts, equalisers and coequalisers. All (co)limits can be constructed from (co)products and (co)equalisers, hence $\mathfrak{Topc}$ has all limits and colimits. ∎

Another way to see the results in Corollary 4.4 is that, by Proposition 4.2, $\mathfrak{Topc}$ is equivalent to the functor category [**Set**, **SupL**]. We know that the limits and colimits in the functor category are calculated point-wise — this is exactly how we have constructed the biproducts, equalisers and coequalisers before. Hence, $\mathfrak{Topc}$ inherits the limit and colimit structures from **SupL**.

This makes it very easy for us to describe the other limit and colimit constructions in $\mathfrak{Topc}$. For example, the pullback of two morphisms $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{C} \to \mathcal{B}$ in $\mathfrak{Topc}$ is given as follows: For any set X,

$$(\mathcal{A} \times_{\mathcal{B}} \mathcal{C})_X = \{ (A, C) \in \mathcal{A}_X \times \mathcal{C}_X \mid FA = GC \}.$$

It is easy to verify that $\mathcal{A} \times_{\mathcal{B}} \mathcal{C}$ is indeed a topological category, from the fact that F, G preserves joins. We have two natural projections

$$p_0 : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{A}, \quad p_1 : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{C},$$

which is universal among morphisms into $\mathcal{A}$ and $\mathcal{C}$ commuting with F, G. We can also use the concrete descriptions of products and equalisers to see this, because the pullback could be realised as the equaliser of the following two morphisms,

$$\mathcal{A} \oplus C \underset{G\circ\pi_C}{\overset{F\circ\pi_C}{\rightrightarrows}} \mathcal{B}$$

The construction of equalisers and pullbacks in $\mathfrak{Topc}$ could potentially be used to create new examples of category of semantics of modal logic, with different properties satisfied by the modality.

4.3.3 Powers and 2-Limits

Now as we have seen, $\mathfrak{Topc}$ is a 2-category. It means the (co)limit structures in $\mathfrak{Topc}$ is richer than ordinary categories, in that it also supports the notion of 2-limits and 2-colimits. For a detailed description of limits in 2-categories see[58]; or see the more general description of limits in enriched categories[52].

For us, we first consider a particularly important type of 2-limits, that of *powers*, or *cotensors*. Roughly speaking, taking powers is a general type of constructions that behaves like function spaces. Since $\mathfrak{Topc}$ is enriched over **SupL**, viz. the 2-categorical structure in $\mathfrak{Topc}$ is posetal, we only need to consider powers over a poset P. Explicitly, for any topological category $\mathcal{A}$, the power of $\mathcal{A}$ over a poset P is an object $\mathcal{A}^P$ in $\mathfrak{Topc}$, such that for any other topological category $\mathcal{B}$ we have the following natural equivalence,

$$\mathfrak{Topc}(\mathcal{B}, \mathcal{A}^P) \cong \mathbf{Pos}(P, \mathfrak{Topc}(\mathcal{B}, \mathcal{A})),$$

where **Pos** is the category of posets. However, since $\mathfrak{Topc}$ is equivalent to the functor category [**Set**, **SupL**], we first show that **SupL** has all powers of posets:

Lemma 4.6: **SupL** is closed under taking powers of posets, i.e. for any suplattice A and any poset P, the power object A^P exists, given by the following formula,

$$A^P = \mathbf{Pos}(P, A).$$

Proof First, notice that $\mathbf{Pos}(P, A)$ is indeed a suplattice under point-wise joins. We show that for any B in **SupL**, the following isomorphism holds,

$$\mathbf{SupL}(B, \mathbf{Pos}(P, A)) \cong \mathbf{Pos}(P, \mathbf{SupL}(B, A)).$$

For any element f in $\mathbf{SupL}(B, \mathbf{Pos}(P, A))$, obviously we have the following mapping,

$$f \mapsto \lambda p \lambda b. f b p.$$

On the other hand, for any g in $\mathbf{Pos}(P, \mathbf{SupL}(B, A))$, we can also associate it with the following function,

$$g \mapsto \lambda b \lambda p.gpb.$$

We leave the readers to see that the two mappings are well-defined function of the above two sets, and that they are mutually inverse to each other. ∎

Then just as all the other limit and colimit constructions in $\mathfrak{Topc}$, powers in $\mathfrak{Topc}$ could also be constructed point-wise: For any topological category $\mathcal{A}$ and any poset P, the power object $\mathcal{A}^P$ exists, and is described by the following formula fibre-wise for any set X:

$$(\mathcal{A}^P)_X = \mathbf{Pos}(P, \mathcal{A}_X) = (\mathcal{A}_X)^P.$$

The above formula indeed defines $\mathcal{A}^P$ as a topological category: Since $\mathcal{A}_X$ is a suplattice for any set X, $\mathbf{Pos}(P, \mathcal{A}_X)$ is also a suplattice under point-wise joins. The fibre-connection $f_! : \mathcal{A}_X \to \mathcal{A}_Y$ for any function $f : X \to Y$ induces the following fibre-connection by post-composing with $f_!$:

$$f_! \circ (-) : \mathbf{Pos}(P, \mathcal{A}_X) \to \mathbf{Pos}(P, \mathcal{A}_Y).$$

From the fact that $f_!$ preserves arbitrary joins it follows easily that so does $f_! \circ (-)$. Hence, $\mathcal{A}^P$ is a well-defined topological category. In fact, a more simple way to see this is that, for any poset P, the following defines an endo-functor on $\mathbf{SupL}$,

$$\mathbf{Pos}(P, -) : \mathbf{SupL} \to \mathbf{SupL}.$$

Thus, by the equivalence of $\mathfrak{Topc}$ with $[\mathbf{Set}, \mathbf{SupL}]$, post-composing with $\mathbf{Pos}(P, -)$ will creates a functor from $\mathfrak{Topc}$ to itself. This amounts exactly to the above explicit formula on fibres. Then it is well-known that, in a functor 2-category, just as $\mathfrak{Topc}$ as $[\mathbf{Set}, \mathbf{SupL}]$, powers, or more generally all 2-limits, are also computed point-wise (cf.[52]). It is also easy to verify directly, but we omit it here for limited space.

The power construction in $\mathfrak{Topc}$ then gives us powerful machinery to construction new topological categories from old ones, where all the fibres of the new topological category are certain function spaces of the old one. Among all the powers, of particular interest is the power of $\mathbf{2}$, the poset $\{0 < 1\}$, which is also the free suplattice on 1 generators. The power $\mathcal{A}^{\mathbf{2}}$ of any topological category $\mathcal{A}$ has fibres of the following explicitly description:

$$\mathcal{A}_X^{\mathbf{2}} = \{\, (A, A') \in \mathcal{A}_X \times \mathcal{A}_X \mid A \leq A' \,\}.$$

More generally, powers of **2** could be used to construct the so-called *comma objects*, which could be understood as certain *lax* version of a pullback. Given any two morphisms $F : \mathcal{B} \to \mathcal{A}$ and $G : \mathcal{C} \to \mathcal{A}$ in $\mathfrak{Topc}$, the comma object $F \downarrow G$ of F and G is a topological category with two projections

$$\pi_{\mathcal{B}} : F \downarrow G \to \mathcal{B}, \quad \pi_{\mathcal{C}} : F \downarrow G \to \mathcal{C},$$

such that there is a 2-cell in $\mathfrak{Topc}$ of the following form, making it the universal one of such 2-cells with the right down corner being F and G,

$$\begin{array}{ccc} F \downarrow G & \xrightarrow{\pi_{\mathcal{C}}} & \mathcal{C} \\ {\scriptstyle \pi_{\mathcal{B}}}\downarrow & \leq & \downarrow{\scriptstyle G} \\ \mathcal{B} & \xrightarrow[F]{} & \mathcal{A} \end{array}$$

Explicitly, the topological category $F \downarrow G$ could be described fibre-wise as follows: For any set X,

$$(F \downarrow G)_X = \{\, (B, C) \in \mathcal{B}_X \times \mathcal{C}_X \mid FB \leq GC \,\}.$$

It is then easy to see that, the power of **2** of a topological category $\mathcal{A}$ is simply the comma object $1_{\mathcal{A}} \downarrow 1_{\mathcal{A}}$. It is also well known that arbitrary comma objects could be constructed using power of **2** and pullbacks (cf.[58]).

These 2-limit structures in $\mathfrak{Topc}$ then creates even more flexibility of general constructions for semantics of modal logic, as we will see in later sections.

At this point, we have already described enough general categorical properties of the category $\mathfrak{Topc}$ of all topological categories, and it is time to turn to the study of questions related to modal logic and our concrete examples of semantics. This will be the topic of the remaining two sections in this chapter.

4.4 The Skeleton of the Landscape

In this section, we will provide a detailed description of the skeleton of the landscape, viz. the full embeddings in Figure 4.1 indicated by the hooked arrows. Since we are know not only concerned with the category of semantics themselves, but the interpretation they provide for modal formulas, we need to take semantic functors into account as well. This suggests that we may give the following definition:

Definition 4.4 (Modal Category): A *modal category* is a topological category $\mathcal{A}$ together with a semantic functor,

$$(-)^+_{\mathcal{A}} : \mathcal{A} \to \mathbf{CABAO}^{\mathrm{op}}.$$

The inter-connection between the information levels then generally consider functors, or simply transformations, between modal categories. However, for the vertical functors to interact well with the interpretation of modal language provided by the modal categories, they will typically satisfy further properties. These properties are usually closely related to the preservation of certain fragments of modal languages we have considered so far. One of the main theme in the remaining two sections in this chapter is then to identify such equivalences.

First, we introduce the general notion of what it means for a functor between two modal categories to preserve certain fragments of modal language:

Definition 4.5 (Preservation of Language): Let $\mathcal{A}, \mathcal{B}$ be two modal categories. Suppose both $\mathcal{A}$ and $\mathcal{B}$ supports the interpretation of certain modal language $\mathcal{L}_0$, then a concrete functor $F : \mathcal{A} \to \mathcal{B}$ between them is said to *preserve the interpretation of the language* $\mathcal{L}_0$, if the following happens: For any object A in $\mathcal{A}_X$ over some set X, for any evaluation function V on X, and for any formula $\varphi \in \mathcal{L}_0$, we have

$$[\![\varphi]\!]^V_A = [\![\varphi]\!]^V_{FA}.$$

When the language $\mathcal{L}_0$ is a multi-agent system for some indexed set Σ, and when the functor F is actually induced by a Σ-indexed product of some basic functor $G : \mathcal{A}_0 \to \mathcal{B}_0$, where $F = G^\Sigma$, $\mathcal{A} = \mathcal{A}_0^\Sigma$ and $\mathcal{B} = \mathcal{B}_0^\Sigma$, we also say that G preserves the interpretation of $\mathcal{L}_0$ iff its Σ-indexed product functor $F = G^\Sigma$ does.

Notice that, here in Definition 4.5 we only consider preservation of languages such that a formula φ is fixed, not with a general linguistic translation of formulas. The reason for this current choice is as follows. First of all, in the context of this thesis, we do not intend to consider different translations of logical connectives other than the modality. Hence, we would want the interpretation of all the other standard propositional connectives to be fixed. Then, our general notion of modal category presented in Definition 4.4

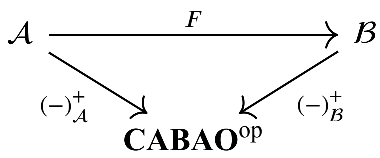
actually provides a semantic approach to possible linguistic translations of modalities, since for the same topological category $\mathcal{A}$ we could consider different semantic functors. Thus, it is crucial to bear in mind that, the notion of modal category not only contains the data of the topological category itself, it also includes the data of a functor from that category to $\mathbf{CABAO}^{\mathrm{op}}$, just like the notion of a concrete category.

In fact, we have shown in Section 3.1.4, like in Example 3.2, that it is certainly possible for a topological category to possess different semantic functors. The notion of a modal functor is then relative to such a choice of semantic functor in the domain and codomain topological categories, making it possible to consider preservation of different modalities from a semantic perspective.

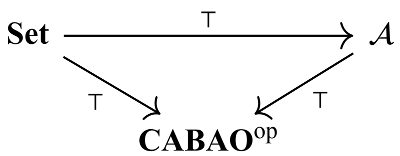
However, it would also be nice to see whether a semantic change of interpretation of modalities could be simulated on the syntactical side. We will consider this more general question in later sections.

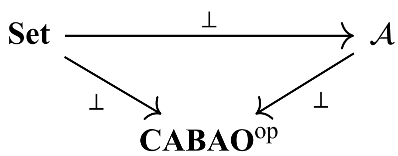
The first and simplest situation we concern is that when the concrete functor between two modal categories commutes with the semantic functors. We call them modal functors:

Definition 4.6 (Modal Functor): A concrete functor $F : \mathcal{A} \to \mathcal{B}$ between two modal categories is called a *modal functor*, if it commutes with the two semantic functors on $\mathcal{A}$ and $\mathcal{B}$, i.e. the following diagramme commutes,

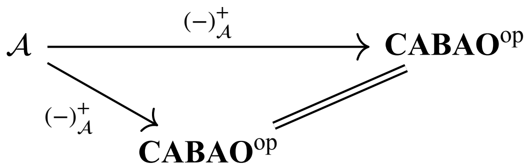


There are actually many examples of modal functors for the modal categories we have considered so far. Trivial examples include several constant semantic functors we have provided in Section 3.1.4. For instance, for any topological category $\mathcal{A}$, the following two diagrammes commute,





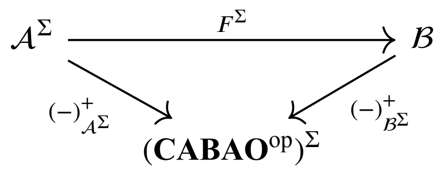
Also, the identity functor on $\mathbf{CABAO}^{\mathrm{op}}$ defines a terminal modal category, in that for any modal category $(\mathcal{A}, (-)^+_{\mathcal{A}})$, there is a trivial commuting diagramme,



This fact means that, the study of modal functors subsumes the study of semantic functors, since every semantic functor on $\mathcal{A}$ is canonically a modal functor from $\mathcal{A}$ to $\mathbf{CABAO}^{\mathrm{op}}$.

Later we will discuss more non-trivial examples of modal functors. In fact, all the functors appearing in the skeleton of the information landscape, viz. all the hooked arrows in Figure 4.1, will be modal functors, as we will show in the future.

Also, if we have a modal functor F between two modal categories $\mathcal{A}, \mathcal{B}$, we can take the Σ-indexed product and obtain the following commuting diagramme,



This means that the treatment of single agent case is essentially the same as the treatment of multi-agent case.

Then we can provide the first example of the main theme we have mentioned above, viz. to characterise modal functors between modal categories using preservation of certain modal languages:

Proposition 4.4: For any concrete functor $F : \mathcal{A} \to \mathcal{B}$ between two modal categories, it is a modal functor, i.e. it commutes with the two semantic functors, iff it preserves the interpretation of formulas in the language $\mathcal{L}^D_\Sigma$, or equivalently, the poorer languages $\mathcal{L}_\Sigma$, or even $\mathcal{L}$.

Proof The if case is easy. Suppose F does not commute with the two semantic functors. Then for some object A in $\mathcal{A}_X$ for some set X, $A^+_{\mathcal{A}}$ and $(FA)^+_{\mathcal{B}}$ would not agree. This means that the two operators on $\wp(X)$ does not coincide, which means they must not coincide on some subset $S \subseteq X$. It then implies that the interpretation of the formula $\Box p$, when the evaluation function V assigns the propositional letter p to S, will not be the same with the semantics provided by A, and the one provided by FA. Hence, if

F preserves the interpretation of the language $\mathcal{L}_\Sigma^D$ (or really, just $\mathcal{L}$), then F must be a modal functor.

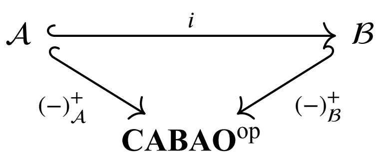
On the other hand, the proof of the only if direction is obviously by induction on the structure of formulas. The only interesting case is the one involving modalities and the dependence atoms. However, since F is assumed to be a modal functor, for any $a \in \Sigma$ we must have

$$(A_a)^+_{\mathcal{A}} = (FA_a)^+_{\mathcal{B}},$$

which means that A_a induces the same operator m_a on $\wp(X)$ through $(-)^+_{\mathcal{A}}$, as for FA_a through $(-)^+_{\mathcal{B}}$. This suggests that the interpretation of the modalities are identity. It also implies that F preserves the interpretation of all dependence atoms, since according to definition, the interpretation of the dependence atoms only relies on the operators on the underlying set. ∎

From a logical perspective, Proposition 4.4 suggests that modal functors are particularly interesting for us to consider, in that they are exactly those transformations that preserve the interpretation of the basic modal logic $\mathcal{L}$ and its multi-agent counterpart $\mathcal{L}_\Sigma$, with the possibly of adding dependence atoms between individual agents, extending the language into $\mathcal{L}_\Sigma^D$.

Many of our examples will actually be much nicer than a bare modal functor. All the functors in the skeleton would actually be fully faithful embeddings between topological categories, that commutes with the canonical semantic functors on them described in Section 3.1. Also, the semantic functors themselves will also be full embeddings. Hence, the exemplar case for us is indicated in the following diagramme of a generic form,



This suggests that the general description of factorisation of full embedding between topological categories could be used to study further properties of such functors. This will be the topic of next section.

We now end this section by describing all the concrete arrows in the skeleton of the information landscape presented by Figure 4.1. From previous discussions, like in Example 2.2 and Example 4.1, we are already familiar with many of them. The goal of the following part is to provide a more detailed description of them being *modal embeddings*, i.e. fully faithful concrete functors that *commute with the semantic functors*, and also address the remaining embeddings not covered before. We start with the top part and left wing of Figure 4.1:

Example 4.2 (Modal Embeddings among **Eqv**, **Pre** and **Kr**): The description of the chain of embeddings among the three relational categories **Eqv** $\hookrightarrow$ **Pre** $\hookrightarrow$ **Kr** is easy. From Example 2.2 (ii)., we know that both **Eqv** $\hookrightarrow$ **Pre** and **Pre** $\hookrightarrow$ **Kr** are full embeddings. From Section 3.1.1, the way we have defined the semantic functors on **Pre** and **Eqv** is through the restriction of the semantic functor on **Kr** to these two full subcategories. Thus, the semantic functors on **Eqv**, **Pre** and **Kr** obviously commute, and these full embeddings are automatically modal. ◂

Part of the left wing of Figure 4.1 is already covered by the embedding **Pre** $\hookrightarrow$ **Kr** described above. We complete the left wing by providing a detailed description of the embedding from **Kr** into **LMon**:

Example 4.3 (Modal Embedding from **Kr** to **LMon**): For any relation (X, R) based on a set X, we associate it an LM-neighbourhood frame (X, E_R) as follows: For any $x \in X$ and $S \subseteq X$,

$$xE_RS \Leftrightarrow R[x] \subseteq S.$$

This definition makes (X, E_R) evidently monotone and left exact. For any function $f : X \to Y$, it is monotone for the relation (X, R) and (Y, Q) iff for any $x \in X$,

$$\exists_f(R[x]) \subseteq Q[fx] \Leftrightarrow R[x] \subseteq f^{-1}(Q[fx]).$$

On the other hand, f is a morphism between the two associated LM-neighbourhood frames (X, E_R) and (Y, E_Q), iff for any $x \in X$ and any $T \subseteq T$,

$$fxE_QT \Rightarrow xE_Rf^{-1}T.$$

By definition, fxE_QT iff $Q[fx] \subseteq T$, and $xE_Rf^{-1}T$ iff $R[x] \subseteq f^{-1}T$. Obviously, the above holds for any T exactly when $R[x]$ is a subset of the smallest choice of T possible, i.e. when $R[x] \subseteq f^{-1}(Q[fx])$. It follows that the above procedure of associating an LM-neighbourhood frame to a relation is functorial, and the resulting functor **Kr** $\hookrightarrow$ **LMon** is a full embedding.

We then see that the embedding commutes with the semantic functors we have described in Section 3.1. Though we have not directly mentioned the semantic functor for **LMon** there, it obviously inherits the description of a semantic functor from the one on **Nb**. Suppose the modality associated to a relation (X, R) is m, and the modality associated to the monotone neighbourhood frame (X, E_R) is m'. Then by definition, for any $x \in X$ and any $S \subseteq X$, we have

$$x \in m(S) \Leftrightarrow R[x] \subseteq S \Leftrightarrow xE_RS \Leftrightarrow x \in m'(S).$$

This means that m and m' are indeed the same operator on $\wp(X)$, which implies that the embedding is indeed a modal one, i.e. it commutes with the semantic functors on **Kr** and on **LMon**. ◀

We then start to describe the right wing of Figure 4.1. First, we show how **Pre** can be fully embedded into **Top** in a coherent way:

Example 4.4 (Modal Embedding from **Pre** to **Top**): For any preorder $(X, \leq)$, we associate a topological space $(X, \tau_\leq)$ based on X. We define that a subset of X is open in $\tau_\leq$ iff it is *upward closed*, i.e. $U \in \tau_\leq$ iff $x \in U$ and $x \leq y$ implies $y \in U$, for any $x, y \in X$. It is easy to verify that $\tau_\leq$ is a well-defined topology. This construction is well-known. $\tau_\leq$ is usually called the *Alexandroff topology* for the order, and $(X, \tau_\leq)$ is usually called an *Alexandroff space*. Again, let $f : X \to Y$ be a function. Now f is monotone for two preorders $(X, \leq)$ and $(Y, \leq)$ iff

$$x \leq y \Rightarrow fx \leq fy.$$

On the other hand, f is continuous for the two Alexandroff spaces $(X, \tau_\leq)$ and $(Y, \tau_\leq)$, iff for any upward closed set V in Y, $f^{-1}V$ is also upward closed in X. On the one hand, suppose f is monotone for the two orders, then for any upward closed set V in Y, $f^{-1}V$ would also be upward closed: If $x \in f^{-1}V$ then $fx \in V$; and if $x \leq y$, then $fy \geq fx$, which implies $fy \in V$ because V is upward closed. This means that $f^{-1}V$ is also upward closed, and hence f is a continuous map for the two Alexandroff spaces. On the other hand, suppose f is continuous. Now for any $x \in X$, $\uparrow fx = \{ w \in Y \mid fx \leq w \}$ is obviously upward closed, which means $f^{-1}(\uparrow fx)$ would also be upward closed. By definition, $x \in f^{-1}(\uparrow fx)$. For any $x \leq y$, it follows that $y \in f^{-1}(\uparrow fx)$, and thus $fy \geq fx$. This shows that f is monotone. This shows that the procedure of associating a preorder to its Alexandroff space indeed defines a full embedding **Pre** $\hookrightarrow$ **Top**.

We then show that this full embedding is again modal. Notice that, Alexandroff spaces have a very nice property, in that every point has a smallest open neighbourhood that contains it. Given any $x \in X$ with a preorder $\leq$ on X, it is easy to see that the smallest upward closed set containing x must be $\uparrow x$. Hence, suppose m is the operator associated to the preorder $(X, \leq)$, and suppose m' is the one associated to $(X, \tau_{\leq})$, viz. the topological interior operator associated to $\tau_{\leq}$, for any $x \in X$ and any $S \subseteq X$, by definition we have

$$x \in m'(S) \Leftrightarrow \text{there exists an open set containing } x \text{ inside } S,$$
$$\Leftrightarrow \uparrow x = \leq[x] \subseteq S \Leftrightarrow x \in m(S).$$

This suggest that m and m' are the same modal operator, thus **Pre** $\hookrightarrow$ **Top** is indeed a full modal embedding. ◀

We complete the right wing by describing the full modal embedding from **Top** to **LMon**. Notice that in Example 4.1 we have already shown that there is a full embedding **Top** $\hookrightarrow$ **LMon**. Thus, we only need to see this embedding furthermore commutes with the semantic functors on them:

Example 4.5 (Modal Embedding from **Top** to **LMon**): It is straight forward to see that this embedding **Top** $\hookrightarrow$ **LMon** given in Example 4.1 is modal. For any topological space (X, τ), let m be its associated operator on $\wp(X)$, and let m' be the associated operator on (X, E_τ). By definition, for any $x \in X$ and any $S \subseteq X$, $x \in m(S)$ iff S is a neighbourhood of x, which by definition exactly when $xE_\tau S$, or $x \in m'(S)$. ◀

The final chain of embeddings at the bottom is then automatic:

Example 4.6 (Modal Embedding among **LMon**, **Mon** and **Nb**): By definition, **LMon** is a full subcategory of **Mon**, and **Mon** is a full subcategory of **Nb**. Just as the case for **Pre**, **Eqv** and **Kr**, the definition of the semantic functor on **LMon** and **Mon** is the one restricted from **Nb**. Hence, these full embeddings are again modal. ◀

4.5 Reflective and Coreflexive Arrows

In the previous section, theoretically we have mainly considered the case where we have two modal categories and a concrete functor between them, which commutes with the two semantic functors. We have identified in Proposition 4.4 the equivalence of the transformation functor being modal, and the preservation of the basic modal language $\mathcal{L}$, or its extension $\mathcal{L}_{\Sigma}^{D}$ to the multi-agent case with dependence atoms between individual modalities. If one carefully looks at how the proof of Proposition 4.4 proceeds, one would realise that the equivalence not only holds for two modal categories, viz. topological categories with semantic functors, but for arbitrary concrete categories with concrete functors into $\mathbf{CABAO}^{\mathrm{op}}$, because the codomain $\mathbf{CABAO}^{\mathrm{op}}$ is always topological, and supports the interpretation of dependence atoms. We then have shown that all the concrete hooked arrows appearing in the skeleton of the landscapes are particular embeddings of this modal kind.

As we have carefully analysed in Chapter 3, the structures of topological categories within the domain of a semantic functor furthermore supports the interpretation of group modalities, and the induced dependence between these group agents. Thus, morphisms in the category $\mathfrak{Topc}$ of topological categories must have more to do with the preservation of these richer structures. In this section, in some sense we will focus more on these topological structures, and consider their corresponding fragments of modal language. Concretely, we will describe all the (co)reflection arrows going backwards along the skeleton appearing in Figure 4.1.

As the second example under our main theme of identifying the properties of vertical transformation functors on one hand, and the preservation of certain fragments of modal languages, we show the following extended result:

Proposition 4.5: Let $F : \mathcal{A} \to \mathcal{B}$ be a modal functor between two modal categories. If it preserves arbitrary meets (resp. joins) fibre-wise, i.e. the induced functions on fibres

$$F_X : \mathcal{A}_X \to \mathcal{B}_X,$$

for any set X, is a morphism in **InfL** (resp. **SupL**), then it preserves the interpretation of any formula in the language $\mathcal{L}_{\Sigma_l}^{D}$ (resp. $\mathcal{L}_{\Sigma_r}^{D}$), for any indexed set Σ. The reverse holds when the semantic functor $(-)_{\mathcal{B}}^{+}$ is injective on objects.

Proof We only prove the case for F preserving meets fibre-wise and the preservation of the interpretation of the language $\mathcal{L}^{D}_{\Sigma_l}$. The other case follows completely dually. We already know from Proposition 4.4 that F is a modal functor iff it preserves the interpretation of formulas in $\mathcal{L}^{D}_{\Sigma}$. Thus, the remain is to show that it further preserves the interpretation of $\bigwedge$-combination of group agents iff it preserves meets fibre-wise.

From how the group modality is defined in the $\bigwedge$-combination of groups described in Section 3.3, it is easy to see that F preserves the interpretation of formulas in $\mathcal{L}^{D}_{\Sigma_l}$, iff the following holds for any $(A_a)_{a\in\Sigma}$ in $\mathcal{A}^{\Sigma}$,

$$\left(\bigwedge_{a\in\Sigma} A_a\right)^{+}_{\mathcal{A}} = \left(F \bigwedge_{a\in\Sigma} A_a\right)^{+}_{\mathcal{B}} = \left(\bigwedge_{a\in\Sigma} FA_a\right)^{+}_{\mathcal{B}}.$$

The first equality holds since F is by assumption a modal functor. And since we have assumed the semantic functor $(-)^{+}_{\mathcal{B}}$ to be an embedding, the above holds iff

$$F \bigwedge_{a\in\Sigma} A_a = \bigwedge_{a\in\Sigma} FA_a,$$

which exactly means F preserves meets fibre-wise, since we have quantified over all possible indexed set Σ. ∎

In particular, if the modal functor $F : \mathcal{A} \to \mathcal{B}$ between two modal categories is in fact a morphism in $\mathfrak{Topc}$ (resp. the right adjoint of a morphism in $\mathfrak{Topc}$), then by definition it must preserves fibre-wise joins (resp. meets), thus preserves the interpretation of the language $\mathcal{L}^{D}_{\Sigma_r}$ (resp. $\mathcal{L}^{D}_{\Sigma_l}$). However, the existence of a left or right adjoint required by morphisms in $\mathfrak{Topc}$ is strictly stronger than the mere preservation of meets or joins fibre-wise. Here we record a few concrete examples of this kind:

Example 4.7: Consider the full embedding **Mon** $\hookrightarrow$ **Nb** of the category of monotone neighbourhood frames into arbitrary neighbourhood frames. For any family of monotone neighbourhood relations $\{E_i\}_{i\in I}$ on a set X, it is easy to verify that, both its intersection $\bigcap_{i\in I} E_i$ and its union $\bigcup_{i\in I} E_i$ are again monotone neighbourhood relations. Hence, the embedding **Mon** $\hookrightarrow$ **Nb** preserves the interpretation of both the language $\mathcal{L}^{D}_{\Sigma_l}$ and $\mathcal{L}^{D}_{\Sigma_r}$. However, we have seen in Example 2.6 that **Mon** is not initially closed in **Nb**, in that the initial lift of single structured sources in **Nb** does not coincide with that in **Mon**. Thus, there is no concrete left adjoint of the embedding **Mon** $\hookrightarrow$ **Nb**.

This actually generates more examples of those full subcategories of **Mon** that is closed both under intersection and union of neighbourhood relations, like **LMon**. Another example is the category **DMon** of *decreasing monotone neighbourhood frames*, or simply *DM-neighbourhood frames*. A monotone neighbourhood frame (X, E) belongs to **DMon** iff for any subset $S \subseteq X$, $E^{-1}(S) \subseteq S$. It is easy to see that **DMon** is again closed under both fibre-wise unions and intersections of neighbourhood relations, thus the embedding **DMon** $\hookrightarrow$ **Mon** also preserves the interpretation of both of the languages $\mathcal{L}^D_{\Sigma_l}$ and $\mathcal{L}^D_{\Sigma_r}$. ◀

However, as we've mentioned, the primary situation of vertical connections within the information landscape we have considered so far is a modal embedding between two modal functors. With the presence of Theorem 4.1, this means that the basic components of the information landscape are those full modal embeddings that either have a left or right adjoint. Of course, we have the following corollary:

Corollary 4.5: If $i : \mathcal{A} \hookrightarrow \mathcal{B}$ is a full modal embedding that determines an injection (resp. the right adjoint of a surjection) in $\mathfrak{Topc}$, then it preserves the interpretation of the language $\mathcal{L}^D_{\Sigma_r}$ (resp. $\mathcal{L}^D_{\Sigma_l}$).

Proof By the dual statement of Proposition 4.1 we know that, if i has a concrete left adjoint then it preserves initial sources, and in particular all fibre-wise meets. Then by Proposition 4.5, it preserves the interpretation of the language $\mathcal{L}^D_{\Sigma_l}$. The other statement follows duality. ∎

Remark 4.6: We have seen that the existence of a concrete reflection or coreflection is a stronger condition than the fact that it preserves arbitrary fibre-wise meets and joins. In the language of fibration we have introduced in Section 2.5, and with Proposition 4.2, the existence of left or right adjoint is equivalent to the preservation of fibre-wise meets or joins, plus commuting with initial or final lifts of single structured sources or sinks. This additional commutation with fibre-connections are not reflected in the preservation of the languages $\mathcal{L}^D_{\Sigma_l}$ or $\mathcal{L}^D_{\Sigma_r}$ yet. However, as we will see in Chapter 5, such properties is crucial when we further consider *dynamics* in our language, which allows us to reason in our language about different models than the current one. ◀

On the other hand, suppose we have a modal functor $F : \mathcal{A} \to \mathcal{B}$ which admits a concrete left adjoint L or a concrete right adjoint R. In general, L, R though will be concrete, they will not be a modal functor *for the current choice of semantic functors on $\mathcal{A}$ and $\mathcal{B}$*! Hence, by Proposition 4.4, they do not preserve the interpretation of even the basic modal language $\mathcal{L}$. However, as we have discussed in Section 4.4, we could certainly change the semantic functor on either $\mathcal{A}$ or $\mathcal{B}$ to make the left adjoint L or right adjoint R as a modal functor. For example, suppose we have an adjunction $L \dashv F$, then let $T = F \circ L$ be the functor part of the induced monad on $\mathcal{B}$. We then have the following commuting diagramme,

$$\begin{array}{ccc} \mathcal{A} & \xleftarrow{\quad L \quad} & \mathcal{B} \\ & {\scriptstyle (-)^+_{\mathcal{A}}} \searrow \quad \swarrow {\scriptstyle (-)^+_{\mathcal{B}} \circ T} & \\ & \mathbf{CABAO}^{\mathrm{op}} & \end{array}$$

This is due to the fact that F is a modal functor for $(-)^+_{\mathcal{A}}$ and $(-)^+_{\mathcal{B}}$,

$$(-)^+_{\mathcal{A}} \circ L = (-)^+_{\mathcal{B}} \circ F \circ L = (-)^+_{\mathcal{B}} \circ T.$$

Thus, by changing the semantic functor on $\mathcal{B}$ from $(-)^+_{\mathcal{B}}$ to $(-)^+_{\mathcal{B}} \circ T$, we have made L into a modal functor as well. This supports our general claim in Section 4.4 that it is enough from a semantic perspective to only consider preservation of languages, rather than more general translations.

However, it would also be interesting to study *whether such semantic transformation could be simulated on a syntactical level*. This leads us to consider the following more general definition of the interaction between model transformation and syntactical translation:

Definition 4.7 (Compatible Translation and Transformation): Given two modal categories $(\mathcal{A}, (-)^+_{\mathcal{A}})$ and $(\mathcal{B}, (-)^+_{\mathcal{B}})$, suppose they both support the interpretation of certain fragment of modal language $\mathcal{L}_0$. A concrete functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *compatible* with a syntactic translation $T : \mathcal{L}_0 \to \mathcal{L}_0$ if the following holds: For any formula $\varphi \in \mathcal{L}_0$, for any object $A \in \mathcal{A}_X$ over some set X, and for any interpretation function V on X, the following holds,

$$[\![T\varphi]\!]^V_A = [\![\varphi]\!]^V_{FA}.$$

Definition 4.7 is could be even more general. We can even talk about the translation between different fragments of modal languages which $\mathcal{A}$ and $\mathcal{B}$ support. However, as we have mentioned in the previous section, at least in the scope of this thesis, we would only like to consider the case where the modal language will be the same, and where the translation T should be inductively generated which fixes the interpretation of all logical connectives other than the modality itself. Also notice that, the preservation of interpretation given in Definition 4.5 can be seen as a special case of the general compatibility of a translation with a transformation, by letting the translation T to be the identity translation.

Now for the previous case with the adjunction $L \dashv F$ and F being a modal functor from $(\mathcal{A}, (-)^+_{\mathcal{A}})$ to $(\mathcal{B}, (-)^+_{\mathcal{B}})$, we could then say that the change of the semantic functor from $(-)^+_{\mathcal{B}}$ to $(-)^+_{\mathcal{B}} \circ T$ where $T = F \circ L$ can be simulated by a translation with respect to a certain fragment of modal language $\mathcal{L}_0$, if there there is translation $T : \mathcal{L}_0 \to \mathcal{L}_0$ compatible with the identity functor on $\mathcal{B}$, but with the domain and codomain $\mathcal{B}$ equipped with the semantic functors $(-)^+_{\mathcal{B}}$ and $(-)^+_{\mathcal{B}} \circ T$, respectively. This is equivalent with the translation T being compatible for the concrete left adjoint $L : \mathcal{B} \to \mathcal{A}$, with $\mathcal{A}, \mathcal{B}$ both equipped with the original semantic functor, because F is modal and commutes with $(-)^+_{\mathcal{A}}$ and $(-)^+_{\mathcal{B}}$.

Below we describe all the reflections and coreflections presented in the Figure 4.1, viz. those arrows going backwards along the skeleton. We will show that relevant modal embeddings we have described in Section 4.4 do have a concrete reflection or coreflection. This could be seen by the explicit description of fibre-connections in Section 2.5 and the meets and joins in the fibres in Section 3.3. We will then explicitly give the construction of the concrete left or right adjoint, and discuss in certain examples possible translations compatible with these reflections and coreflections.

Example 4.8 (The Reflection and Coreflection from **Pre** to **Eqv**): We consider the embedding **Eqv** $\hookrightarrow$ **Pre**. From the description of final and initial lifts of single structured source and sink in **Pre** and **Eqv** in Example 2.4, both of them coincide in **Pre** and in **Kr**. Thus, the embedding **Eqv** $\hookrightarrow$ **Pre** preserves them. Furthermore, for any set X, the induced embedding $\mathbf{Eqv}_X \hookrightarrow \mathbf{Pre}_X$ preserves both arbitrary meets and joins. In Example 3.7, we have shown that meets in both of the fibres are computed by intersection; the transitive closure of the union of a family of equivalence relation is in fact also an equivalence relation. Such facts implies that $\mathbf{Eqv}_X \hookrightarrow \mathbf{Pre}_X$ preserves both meets and joins. Combining all this, it follows that the embedding **Eqv** $\hookrightarrow$ **Pre** preserves both initial sources and final sinks, and thus by Proposition 4.1 it has both a concrete right and left adjoint, which makes **Eqv** simultaneously a reflexive and coreflexive subcategory of **Pre**. They consist of the back and forth arrows in $\mathfrak{Topc}$ presented in Figure 4.2.

Now we can easily describe what the left and right adjoints are, simply by taking the fibre-wise right and left adjoint for the inclusion $\mathbf{Eqv}_X \hookrightarrow \mathbf{Pre}_X$, according to the proof of Proposition 4.1. The reflection, or the left adjoint, takes a preorder $(X, \leq)$ to the least equivalence relation $\leq^\sharp$ generated by it,

$$\leq^\sharp = \bigcap \{ \sim \in \mathbf{Eqv}_X \mid \leq \subseteq \sim \}.$$

We have noted that arbitrary intersection of equivalence relations on a set is again an equivalence relation, thus the above formula gives a well-defined equivalence relation, which is obviously the smallest one containing $\leq$.

The coreflection, or the right adjoint, takes it to the largest equivalence relation $\leq^\flat$ it contains. Explicitly, for any $x, y \in X$ we have

$$x \leq^\flat y \Leftrightarrow x \leq y \,\&\, y \leq x.$$

In the literature of preference logic, this is known as the *indifference relation*[17-18]. We left the readers to see that the two assignments are functorial, and they indeed form the concrete left and right adjoint for the full embedding $\mathbf{Pre} \hookrightarrow \mathbf{Eqv}$. ◀

Example 4.9 (The Reflection from **Kr** to **Pre**): Similar to Example 4.8, the embedding $\mathbf{Pre} \hookrightarrow \mathbf{Kr}$ is reflexive. For any set X, we know that the fibre-wise embedding $\mathbf{Pre}_X \hookrightarrow \mathbf{Kr}_X$ preserves arbitrary meets, and from Example 2.4 we know that the initial lift of a single structured source in **Pre** is inherited from **Kr**. This means that the embedding $\mathbf{Pre} \hookrightarrow \mathbf{Kr}$ preserves all initial sources, and thus by the dual statement of Proposition 4.1 again, it has a reflection.

For any object (X, R) in **Kr**, its reflection in **Pre** is simply given by the reflexive and transitive closure R^* of R. Again, it is easy to see that such an assignment is functorial, and it is indeed the concrete left adjoint of the embedding $\mathbf{Pre} \hookrightarrow \mathbf{Kr}$.

However, the embedding $\mathbf{Pre} \hookrightarrow \mathbf{Kr}$ does not has a concrete right adjoint. This can be seen from the fact that, fibre-wise, the embedding $\mathbf{Pre}_X \hookrightarrow \mathbf{Kr}_X$ does not preserves joins. For any family of preorders, the join of them is the *transitive closure* of their union, while their join in **Kr** is simply given by the union. The discrepancies are already present in the discussion of group knowledge in Example 3.7, and this is what makes the notion of common knowledge distinct from the knowledge known by everyone. The final lifts of single structured sinks in **Pre** also differs from that in **Kr**, as we have also shown in Example 2.4.

Quite unusually, but well-known in the literature of modal logic, in this case there is indeed a compatible syntactic translation with respect the reflection **Kr** → **Pre**, for the infinitary fragment of modal logic. We use $\mathcal{L}_\infty$ to denote the language extending the basic modal language $\mathcal{L}$ with infinitary conjunctions and disjunctions. Then we inductively define a translation $T : \mathcal{L}_\infty \to \mathcal{L}_\infty$, by fixing all the other logical connectives, but sending a modalised formula as follows,

$$T(\Box\varphi) \equiv \bigwedge_{n=0}^{\infty} \Box^n T(\varphi),$$

where $\Box^0 \varphi \equiv \varphi$ and $\Box^{n+1}\varphi \equiv \Box\Box^n \varphi$. Now by a simply inductive argument we can show that, for any formula φ, any relation R on X, and any evaluation function V on X, the following holds,

$$[\![T\varphi]\!]^V_R = [\![\varphi]\!]^V_{R^*}.$$

This function T then provides a well-defined compatible syntactical translation for the reflection from **Kr** to **Pre**, according to Definition 4.7. It is also evident from the construction of transitive closure that the use of infinitary conjunction here is essential. ◀

Example 4.10 (The Coreflection from **LMon** to **Kr**): Let's now consider the remaining embedding **Kr** ↪ **LMon** in the left wing. Recall from Example 4.3, the embedding takes any Kripke frame (X, R) to the LM-neighbourhood frame (X, E_R), with E_R defined as follows: For any $x \in X$ and $S \subseteq X$,

$$xE_RS \Leftrightarrow R[x] \subseteq S.$$

Now suppose we have a family of relations $\{R_i\}_{i\in I}$, we know that

$$xE_{\bigcup_{i\in I} R_i}S \Leftrightarrow \bigcup_{i\in I} R_i[x] \subseteq S \Leftrightarrow \forall i \in I[xE_{R_i}S].$$

This implies that

$$E_{\bigcup_{i\in I} R_i} = \bigcap_{i\in I} E_{R_i}.$$

Now from Example 2.3 (v)., we know that the canonical order in the topological category **LMon** as a full subcategory of **Nb** is the *reverse* of inclusion, and from Example 4.1 we also know that the joins in each fibre $\mathbf{LMon}_X$ are simply calculated as intersections of neighbourhood relations. This means that, the embedding $\mathbf{Kr} \hookrightarrow \mathbf{LMon}$ preserves joins fibre-wise.

For the final lifts of single structured sources, let $f : X \to Y$ be an arbitrary function, and let R be a relation on X. By definition, for any $y \in Y$ we have

$$f_! R[y] = \bigcup_{x \in f^{-1}(y)} \exists_f R[x] = \exists_f \bigcup_{x \in f^{-1}(y)} R[x],$$

which means that, for any subset $S \subseteq Y$,

$$f_! R[y] \subseteq S \Leftrightarrow \bigcup_{x \in f^{-1}(y)} R[x] \subseteq f^{-1}(S).$$

Both of the two steps of reasoning use the fact that $\exists_f$ is left adjoint to f^{-1}. On the other hand, from Example 4.1 we also know that **LMon** is finally closed in **Mon**, and recall from Example 2.6 the final lift of E_R along f in **Mon**, hence in **LMon**, is given as follows: For any $y \in Y$ and any $S \subseteq Y$,

$$\begin{aligned} y f_! E_R S &\Leftrightarrow \forall x \in f^{-1}(y)[x E_R f^{-1}(S)] \\ &\Leftrightarrow \forall x \in f^{-1}(y)[R[x] \subseteq f^{-1}(S)] \Leftrightarrow \bigcup_{x \in f^{-1}(y)} R[x] \subseteq S. \end{aligned}$$

Hence, we indeed have

$$f_! E_R = E_{f_! R},$$

which means the construction also commutes with final lifts of single structured source. Combining above, by Proposition 4.1, this embedding has a concrete right adjoint.

The coreflection can be described explicitly as follows. For any LM-neighbourhood frame (X, E), let $E^\flat$ denote the following neighbourhood relation: For any $x \in X$ and $S \subseteq X$,

$$x E^\flat S \Leftrightarrow \bigcap E[x] \subseteq S.$$

This way, $E^\flat$ corresponds to a relation R_E, such that for any $x \in X$,

$$R_E[x] = \bigcap E[x].$$

Again, we left the readers to see that this gives us a well-defined coreflection from **LMon** to **Kr**. ◀

We now start to describe the right wing of Figure 4.2, by first considering the coreflection from **Top** to **Pre**:

Example 4.11 (The Coreflection from **Top** to **Pre**): We prove the existence of a coreflection by directly provide a right adjoint of the full modal embedding from **Pre** to **Top** given in Example 4.4. In fact, we have already encountered this right adjoint functor in Example 2.2 (iv). of Chapter 2. Recall from there that for any topological space (X, τ), we associate it with the specialisation relation $\preceq$ defined as follows: For any $x, y \in X$, we have

$$x \preceq y \Leftrightarrow \forall U \in \tau[x \in U \Rightarrow y \in U].$$

We prove that this indeed gives the right adjoint of the modal embedding $\mathbf{Pre} \hookrightarrow \mathbf{Top}$ we have described in Example 4.4 by showing the following natural isomorphism for any topological space (X, τ) and any preorder $(Y, \leq)$:

$$\mathbf{Top}((Y, \tau_{\leq}), (X, \tau)) \cong \mathbf{Pre}((Y, \leq), (X, \preceq)).$$

For any function $f : Y \to X$, first suppose it is a continuous function from $(Y, \tau_{\leq})$ to (X, τ). Then for any $y \leq y'$ in Y, we show that $fy \preceq fy'$. Suppose there exists an open neighbourhood U of fy. Then since f is continuous, $f^{-1}(U)$ is open in $(Y, \tau_{\leq})$, which means it is upward closed. Then we have the following chain of implication,

$$fy \in U \Rightarrow y \in f^{-1}(U), \quad y \leq y' \Rightarrow y' \in f^{-1}(U) \Rightarrow fy' \in U.$$

This suggests that U also contains fy', thus by definition of the specialisation order on X, we have $fy \preceq fy'$. This proves that f is monotone for the two preorders.

On the other hand, suppose f is a monotone function from $(Y, \leq)$ to $(X, \preceq)$. We show that f is also continuous for the two topological spaces. Given any open set U in X, we prove $f^{-1}(U)$ is upward closed. Suppose $y \in f^{-1}(U)$, i.e. $fy \in U$. Then for any $y' \geq y$, by monotonicity we have $fy' \succeq fy$, which by definition means that U also contains fy', or equivalently, $y' \in f^{-1}(U)$. This proves that $f^{-1}(U)$ is upward closed, and thus an open set in $(Y, \tau_{\leq})$, which ultimately shows the continuity of the function f between the two topological spaces. This completes the proof that we have the above isomorphism between the two hom-sets in **Pre** and **Top**, and we leave naturality of such isomorphisms for the readers to check. ◀

Now we have already covered the description of the right adjoint of the modal embedding from **Top** to **LMon** in Example 4.1, when we discuss the factorisation in $\mathfrak{Topc}$. Thus, we now what the right wing consists of in Figure 4.2. We are left with the bottom two reflection between certain full subcategories of **Nb**:

Example 4.12 (The Reflection from **Nb** to **Mon** and **Mon** to **LMon**): Now in Example 2.6, Example (v). and in Example 4.1, we have already mentioned that, though **Mon** and **LMon** are both closed under arbitrary intersection and union of neighbourhood relations in the fibre, only the final lifts of single structured sinks in **Mon** and **LMon** coincide with that in **Nb**. This suggests that both of the modal embedding **LMon** $\hookrightarrow$ **Mon** and **Mon** $\hookrightarrow$ **Nb** preserve all final sinks, and thus have concrete right adjoints.

We first describe the reflection from **Nb** to **Mon**. If you recall how we have shown the equivalence of categories between **Mon** and **Evi** in Remark 2.2, it should be obvious that for any neighbourhood frame (X, E), the associated monotone neighbourhood frame (X, E_m) should be as follows: For any $x \in X$ and $S \subseteq X$,

$$xE_mS \Leftrightarrow \exists T \subseteq S[xET].$$

By definition, E_m would be monotone.

Now given a monotone neighbourhood frame (X, F), the associated LM-neighbourhood frame (X, F_{lm}) should be described as below: For any $x \in X$ and $S \subseteq X$, $xF_{lm}S$ iff there exists a finite family of subsets $\{S_i\}_{i=1}^n$, such that $S = \bigcap_{i=1}^n S_i$ and xFS_i for all $1 \leq i \leq n$. It is also obvious that F_{lm} is an LM-neighbourhood frame — notice that the case for $n = 0$ recovers $xF_{lm}X$ for any $x \in X$.

The universal description for all concrete reflections of a full embedding between two topological spaces is always that, an object in the lower level is sent to the least object in the higher level that contains the original object. We leave for the readers to verify that the above two constructions are indeed of this kind. ◀

4.6 Other Structures in the Information Landscape

Up to this point, we have already completed the description of all the arrows in Figure 4.1, and have ultimately seen that they constitute morphisms in $\mathfrak{Topc}$, organising themselves into the diagramme presented in Figure 4.2. We end this chapter by discussing some other structures in the information landscape, applying the general theory of categorical structures in $\mathfrak{Topc}$ developed in the first three sections of this chapter.

4.6.1 Image Factorisation and Beck-Chevellay Condition

In $\mathfrak{Topc}$, or more restrictively for our choice of the corner of the information landscape presented in Figure 4.2, we may have various morphisms which can be composed. Then a general and natural question is then, which types of objects in the target actually come from those in the source along this morphism. Mathematically, this asks for the image factorisation of morphisms in $\mathfrak{Topc}$, which we have described in Section 4.2.

There is a very nice example already in Figure 4.2. Though we have described every morphism in $\mathfrak{Topc}$ appearing in Figure 4.2 representing the landscape of information, one missing part is the discussion of the central square in Figure 4.2 as follows,

$$\begin{array}{ccc} \mathbf{Pre} & \rightarrowtail & \mathbf{Top} \\ \twoheaduparrow & & \twoheaduparrow \\ \mathbf{Kr} & \rightarrowtail & \mathbf{LMon} \end{array}$$

The first observation is that it is a commutative in $\mathfrak{Topc}$:

Lemma 4.7: The above square commutes in $\mathfrak{Topc}$.

Proof Given any Kripke frame (X, R), the reflection from **Kr** to **Pre** takes it to (X, R^*), where R^* is the reflexive and transitive closure of R. The Alexandroff topology associated with (X, R^*) then treats all R^*-closed subsets as opens. The crucial observation is that, for any subset $S \subseteq X$, it is R^*-closed iff it is R-closed. This is because R^* could be explicitly described as follows: For any $x, x' \in X$, xR^*x' iff there exists a finite number of elements $x_1, \cdots, x_n$ in X, such that $x_i R x_{i+1}$ for any $1 \leq i \leq n-1$, and that $x = x_1$ and $x' = x_n$. Thus, the topology associated to (X, R) going this way has R-closed subsets as opens.

Going the other way around, recall that the LM-neighbourhood frame (X, E_R) associated to (X, R) is as follows: For any $x \in X$ and $S \subseteq X$,

$$xE_RS \Leftrightarrow R[x] \subseteq S.$$

Now the reflection from **LMon** to **Top** takes E_R on X to the topology as follows,

$$\tau_{E_R} = \{\, S \subseteq X \mid S \subseteq E_R^{-1}(S) \,\}.$$

In other words, a subset S is open iff for any $x \in S$, xE_RS, or equivalently, $R[x] \subseteq S$. This exactly means that S is R-closed. Hence, it generates the same topology as before, and thus the diagramme commutes. ∎

There are two ways to read this commuting diagramme. On one hand, the commuting diagramme identifies a morphism in $\mathfrak{Topc}$ from **Kr** to $\mathfrak{Topc}$. It is the one we have constructed in the above proof of Lemma 4.7: It takes an arbitrary Kripke frame (X, R) to its associated topological space (X, τ_R), where τ_R identifies open sets as those R-closed subsets in X. Then the following composition in $\mathfrak{Topc}$

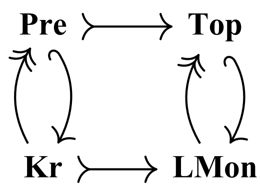
$$\mathbf{Kr} \twoheadrightarrow \mathbf{Pre} \rightarrowtail \mathbf{Top},$$

could be viewed as the image factorisation of the morphism $\mathbf{Kr} \to \mathbf{Top}$ in $\mathfrak{Topc}$, which identifies **Pre** as the image of this morphism. It means that, to consider those topological spaces that comes from a Kripke frame as above, it suffices to only consider the case where the relation is a preorder. In fact, Alexandroff topological spaces could be characterised intrinsically as those topologies closed under both arbitrary unions and intersections. And it is an easy exercise to prove the above claim using this characterisation.

On the other hand, the above diagramme also identifies **Pre** as a subquotient of **LMon**, with the intermediate stage **Kr**. As we've mentioned in Section 4.2, this constitutes the factorisation of the full embedding $\mathbf{Pre} \hookrightarrow \mathbf{LMon}$.

However, the above diagramme has more structure than those presented by the above two ways of reading it, in that it further satisfies the following property:

Lemma 4.8: The above diagramme also satisfies the so-called *Beck-Chevalley condition*, i.e. the two injections not only commute with the two surjections, but also their right adjoints:



Proof Consider a preorder $(X, \leq)$. According to the description of modal embeddings in Section 4.4, the associated LM-neighbourhood frame (X, E_1) by viewing it as a general Kripke frame is given as follows: For any $x \in X$ and $S \subseteq X$,

$$xE_1S \Leftrightarrow \leq[x] = \uparrow x \subseteq S.$$

On the other hand, the associated LM-neighbourhood frame (X, E_2) by viewing it as a topological space through the Alexandroff topology is as below: For any $x \in X$ and $S \subseteq X$,

$$xE_2S \Leftrightarrow \exists U \in \tau_{\leq}[x \in U \subseteq S] \Leftrightarrow \uparrow x \subseteq S.$$

The second equivalence is due to the fact that, by definition, $\uparrow x$ is the smallest open set containing x. Thus, it is now evident that the two neighbourhood frames E_1 and E_2 are the same. ∎

Remark 4.7: It is worthy commenting at this point, that the author has been constantly thinking about the problem of using an order-theoretic approach to study logical systems for a long time. The first paper[59] on this topic shows how our familiar propositional logical connectives can emerge through a very simple consistency relation; and in another paper of the author[60], much more related to the current thesis, the very same category $[\mathbf{Set}, \mathbf{SupL}]$, or in other words, $\mathfrak{Topc}$, is used to serve as a universe, modelling the syntax and consequence relations of arbitrary logical theories with algebraic signatures. There, we have shown that there is a bijection between quotients on an object $\mathcal{A}$ in $[\mathbf{Set}, \mathbf{SupL}]$ and (structural) consequence relations on $\mathcal{A}$. Then the commutative diagrammes of the above form naturally arises when we consider transformations between two logical theories. Moreover, the Beck-Chevalley condition appears as one of the key property of identifying when two logics are biinterpretable, which is crucial for establishing the result about *algebraisation of logic*. It is then surprising for us to see that the same mathematical pattern appears here again.

Moreover, this also suggests that the category $\mathfrak{Topc}$ of topological categories could be used as a common ground where the syntax and semantics of different fragments of modal logic meet, and the *internal structure* of $\mathfrak{Topc}$ could then provide further analysis of general modal logics. However, such a topic is beyond the scope of this thesis, and we leave this for future exploration. ◀

4.6.2 Biproducts and Comma Objects

The categorical structure in $\mathfrak{Topc}$ also allows us to combine and extend the individual mono-modal languages to other bi-modal, or in general n-modal, fragments of modal languages. This is in general different from the multi-agent extension of the basic modal language $\mathcal{L}$, because in the multi-agent case, all modalities are of the same type. In this section, we are considering about combining modalities of different types.

The most simple case is simply using the biproduct structure to combine two types of modalities in an independent way. Given any two modal categories $(\mathcal{A}, (-)^+_{\mathcal{A}})$ and $(\mathcal{B}, (-)^+_{\mathcal{B}})$, if we combine the two semantic functor together, we will get a functor of the following form,

$$(-)^+_{\mathcal{A}} \oplus (-)^+_{\mathcal{B}} : \mathcal{A} \oplus \mathcal{B} \to (\mathbf{CABAO}^{\mathrm{op}})^{\oplus 2}.$$

This could be read as the topological category $\mathcal{A} \oplus \mathcal{B}$ now supports the interpretation of a bi-modal language. Any object (A, B) in the fibre $(\mathcal{A} \oplus \mathcal{B})_X = \mathcal{A}_X \times \mathcal{B}_X$ provides the interpretation of the two types of modalities using $(-)^+_{\mathcal{A}}$ and $(-)^+_{\mathcal{B}}$, respectively. This could be generalised to arbitrary biproducts, and we have already touched upon this even back to Chapter 3 when first introducing semantic functors.

More interestingly, the 2-limit structures in $\mathfrak{Topc}$ allow us to construct new n-modal languages such that there are certain relations between these modalities. One concrete example is the *plausibility models*, which is used in general modal logic literature to model agents' both knowledge and beliefs. Mathematically, a plausibility model is a set X equipped with both an equivalence relation and a preorder, where the equivalence relation represents the agents current epistemic range, while the preorder could be understood as the subjective likelyhood order. Within this interpretation, the preorder and the equivalence relation cannot be unrelated. In fact, we always require that the preorder is contained in the equivalence relation, reflecting the consideration that the agent's subjective likelyhood judgement of the possible situations is always bounded by its epistemic range. For more discussions on plausibility models and the epistemic and doxastic reasoning it supports, consult[8,15].

Let **Pl** be the category of plausibility models, with objects being plausibility models and morphisms being monotone functions for both of the two relations. Using our general description of the construction of comma objects presented in Section 4.3.3, it is easy to see that **Pl** is precisely the comma object of the following two morphisms in $\mathfrak{Topc}$,

$$\begin{array}{ccc} \mathbf{Pl} & \xrightarrow{\pi_1} & \mathbf{Eqv} \\ {\scriptstyle \pi_0}\downarrow & \leq & \downarrow \\ \mathbf{Pre} & = & \mathbf{Pre} \end{array}$$

Concretely, there is a 2-cell for the above diagramme means exactly that, the preorder part of a plausibility model is always contained in the equivalence relation. Moreover, **Pl** is the universal category for this situation; this roughly means that it is exactly the category that contains all such models. This comma object construction, together with the above two projections π_0 and π_1, again means that **Pl** supports the interpretation of a bi-modal language, with one modality being the usual knowledge modality, and the other one is for what's called *safe belief*. From this example, we can see that the 2-limit structures in $\mathfrak{Topc}$ are actually useful when creating new models of modal logic.

We have mentioned in Section 4.3.3 that comma objects could be constructed using pullbacks and powers over **2**. The more general power construction given there could be used to construct other categories of models with more complicated properties. Such infinite possibilities are waiting for more people to explore.

Chapter 5 Horizontal Dynamics within the Landscape

Let's briefly recall that in Chapter 4, besides a very detailed description of the total information landscape in terms of the categorical structure of the large category $\mathfrak{Topc}$ of all topological categories, which is development purely on the semantic side, we have also described how the vertical model transformations connects various syntactic structures of modal logic, including the modalities, modal dependence and group structures, to the semantic structures within topological categories, by proving exact correspondence results like Proposition 4.4 and Proposition 4.5. In this Chapter, we will extend the scope of the last aspects, by also considering *dynamic* extensions of modal logic, trying to provide a satisfactory answer to our last Problem 1.3.

Specifically, we are first going to look at how the various dynamic extensions of modal logic found in the literature, including the mechanisms deployed in PAL, public announcement logic, and DEL, dynamic epistemic logic, could be generalised and described in a uniform way for any topological categories. We call the generalisation of the former *PAL style update*, and of the latter *product type update*. These will be the topics of Section 5.1 and Section 5.2, respectively. In Section 5.3, we will look at special cases of product type update, showing that various types of logical dynamics developed in the literature could be viewed as special cases of our general description of product type update.

In the meantime, we will also consider vertical transformations between topological categories, so as to identify which particular part of the structure of a topological category does a fragment of dynamic extension correspond to. In particular, for all the extensions of modal logic we have described in Chapter 3, we have seen that they correspond to semantic structures *within single fibres* of a topological category, be it the partial order for the modal dependence or the complete lattice operations for the group structures. However, as we have seen in Chapter 2, another crucial part of the structure of a topological category lies in its *fibre connections*, and in this Chapter we will see their close connection with logical dynamics.

We end this chapter by saying a few words how our technical development in this chapter could be viewed as describing a form of horizontal-vertical interactions within the landscape of information.

5.1 PAL-Style Dynamics

To warm up, we first look at the most simple kind of dynamical behaviours that could happen in a topological category, which is a generalisation of the usual dynamics of public announcement we consider in various fragments of modal logic.

Again let us assume $(\mathcal{A}, (-)^+)$ is a modal category, with A being a model in $\mathcal{A}$ over the set X, and V being an interpretation function on X. The announcement with a formula φ in a relevant fragment of modal logic, which we denote the update as $!\varphi$, naturally restricts the domain to the subset $S = [\![\varphi]\!]_A^V$, viz. the set of points where φ holds. This then creates an inclusion map

$$i : S \hookrightarrow X.$$

Now we need to see the updates set of states S as also a model in the modal category $\mathcal{A}$. The natural candidate is the canonically associated object i^*A on S. Intuitively, the initial lift of A along the inclusion i gives out the universal model over the domain S that "maximally makes the additional structure stay the same way as A is" — the quoted statement is made precise mathematically by the universal property of the initial lift along a morphism which we have introduced in Chapter 2. We will see as follows that this universal description of initial lifts along an inclusion map recovers in each concrete case the right types of dynamics we consider in the literature:

Example 5.1 (PAL Style Dynamics in Relational Structures): Recall from Example 2.4 that initial lifts along any single structured source coincide within **Kr**, **Pre** and **Eqv**. When the map $i : S \hookrightarrow X$ is an inclusion, the initial lift in these three category is especially easy to describe. For any set X and any relation (resp. preorder, equivalence relation) R on X, the initial lift of R along i is given as follows: For any $x, y \in S$,

$$x i^* R y \Leftrightarrow x R y.$$

Of course, when writing xRy we have already implicitly viewed x, y as two elements of X along the inclusion map i. In other words, the initial lift i^*R is simply the relation R restricted to the smaller domain S. ◀

Example 5.2 (PAL Style Dynamics in Topological Spaces): Given any topology τ on X, its initial lift $i^*\tau$ on S is the subspace topology on S, explicitly given as follows,

$$i^*\tau = \{ U \cap S \mid U \in \tau \}.$$

If you compare with the description of general initial lift along a single structured source given in Example 2.5, you will realise that the two description indeed coincide in the special case of inclusion maps, because the preimage of any subset $T \subseteq X$ of i is simply the intersection $T \cap S$. ◀

Example 5.3 (PAL Style Dynamics in Neighbourhood Structures): Consider the general description of fibre connection of neighbourhood structures given in Example 2.6 and in Example 4.1, the initial lift of an arbitrary single structured source does not coincide for **Nb**, **Mon** and **LMon**. However, the claim is that they do coincide for inclusion maps specifically. Recall that, given any neighbourhood relation E on X, its initial lift along i is given as follows: For any $x \in S$ and $T \subseteq S$,

$$xi^*ET \Leftrightarrow \exists U \subseteq X[T = U \cap S \,\&\, xEU].$$

It is easy to see that, if we start from a monotone neighbourhood frame E, its initial lift i^*E would also be monotone. Suppose we have xi^*ET, which means there exists $U \subseteq X$ with $T = U \cap S$ and xEU. Then for any $T \subseteq V \subseteq S$, consider the subset $U \cup V$. First we have the following calculation,

$$(U \cup V) \cap S = (U \cap S) \cup (V \cap S) = T \cup V = V.$$

Furthermore, since E is monotone, xEU would imply that $xE(U \cup V)$. Thus, it follows that we also have xi^*EV, witnessed by the subset $U \cup V$. Hence, we no longer to monotonise the initial lift i^*E calculated in **Nb**, which means they coincide in **Nb** and **Mon**. Now by Example 4.1, we know that initial lifts in **LMon** and in **Mon** are calculated in the same way. Thus, the PAL style dynamics also coincide for **Nb**, **Mon** and **LMon**, which is uniformly described by the above formula of initial lifts. ◀

Furthermore, the description of initial lift of single structured source along an inclusion map also applies to the case of evaluation function:

Example 5.4 (PAL Style Dynamics for Evaluation Functions): Recall from Example 2.7 that, there is a topological category of sets equipped with an evaluation function for some fixed set of propositional variables. Now consider any evaluation function V on a set X, recall from the description of bifibrational structure in Example 2.7, its initial lift along the inclusion map i on S is given as follows: For any propositional letter p, we have

$$i^*V(p) = V(p) \cap S.$$

Again this is due to the fact that inverse images of i is calculated as taking intersections with the subset S. This is exactly how we usually specify the evaluation function on the smaller domain S, making it into an actual model of modal logic, so that it supports the interpretation of PAL formulas. It is nice to see here that, from a topology categorical perspective, it accords to the same general description of initial lift along inclusion maps with other information structures. ◀

The informational event of announcing φ has its syntactical representations in the form of dynamic operators of the following two types, $[!\varphi]$ and $\langle !\varphi \rangle$. The dynamic operator is responsible to evaluate any formula ψ in its scope in the newly defined model i^*A over S. However, we would also like to transport the evaluation in the smaller domain S back into the original set X, to conform with the recursive style of dynamic operators in the syntax. The operators on power sets associated to a function — and more generally to a relation — described in Section 3.1.1 is of great help. We will assume the modal logic supports the interpretation of a certain fragment of modal logic $\mathcal{L}_0$. Then the interpretation of PAL style dynamic formulas is defined as follows:

Definition 5.1 (Interpretation for PAL Style Dynamics): Let $\mathcal{L}_0^{\mathbf{PAL}}$ be the extension of $\mathcal{L}_0$ which allows forming formulas of the form $[!\Phi]\Psi$, where we have used the upper case Greek letters to denote the formulas in the extended language $\mathcal{L}_0^{\mathbf{PAL}}$. We adopt the same notation convention with the current literature on dynamic logic and introduce the dual operator as follows,

$$\langle !\Phi \rangle \Psi \;:\equiv \neg [!\Phi] \neg \Psi.$$

To provide the interpretation of formulas in $\mathcal{L}_0^{\mathbf{PAL}}$, it suffices to give the clause of PAL dynamic operators: For any formulas Φ, Ψ in $\mathcal{L}_0^{\mathbf{PAL}}$, any model A in $\mathcal{A}$ over a set X, and any evaluation function V on X, let $i : [\![\Phi]\!]_A^V \hookrightarrow X$ be the inclusion map from the set of Φ-points into the total domain. Then the interpretation of the dynamic formula $[!\Phi]\Psi$

and $\langle !\Phi\rangle\Psi$ is as follows:

$$[\![[!\Phi]\Psi]\!]^V_A := \forall_i [\![\Psi]\!]^{i^*V}_{i^*A}, \quad [\![\langle !\Phi\rangle\Psi]\!]^V_A := \exists_i [\![\Psi]\!]^{i^*V}_{i^*A}.$$

This makes every formula in the extended language $\mathcal{L}_0^{\mathbf{PAL}}$ interpretable in the modal category $\mathcal{A}$.

If we unwrap the definition of the two operators $\forall_i$ and $\exists_i$, we would obtain the following more familiar local version of the semantics of PAL style dynamical logic: For any $x \in X$,

$$A, x \vDash [!\Phi]\Psi \text{ iff } A, x \vDash \Phi \Rightarrow i^*A, x \vDash \Psi,$$
$$A, x \vDash \langle !\Phi\rangle\Psi \text{ iff } A, x \vDash \Phi \,\&\, i^*A, x \vDash \Psi.$$

The (meta linguistic) implication and conjunction appearing in the above two clauses simply correspond to the right and left adjoint behaviour with respect to the inverse image function, as we have defined in Section 3.1.1.

The universal description in Definition 5.1 for any modal category, plus the concrete scenarios of initial lifts along injections given above for all the exemplar categories we have considered in previous chapters, together create a unifying treatment of PAL style dynamics in each case. The fact that there exists a universal description of PAL style logical dynamics in *any* modal category, viz. a topological category with a semantic functor, could also be viewed as a more satisfactory replacement than the existence of tracking operators of PAL stye dynamics between two information levels. Our formulation suggests that any modal category has sufficient internal structures to support an intrinsic notion of PAL update, and that should be the very notion of PAL update we care about in any concrete examples of modal categories. All the type of dynamics we investigate in this chapter are actually of this form.

However, the above description only provides a conceptual unification of the structure of topological categories that pertains to the interpretation of PAL dynamics. But it does not mean that the interpretation of PAL formulas will be preserved by any vertical transformations between two information levels, and that should be the replaced problem to consider about, as we will do immediately.

We can now formulate which types of vertical transformations will preserve the interpretation of PAL style dynamics. As before, we will consider a concrete functor F :

$\mathcal{A} \to \mathcal{B}$ between two modal categories, that already preserves the interpretation of certain fragment of modal logic $\mathcal{L}_0$ that extends the basic modal language $\mathcal{L}$. By Proposition 4.4, we will assume F is a modal functor. We state the necessary and sufficient condition for F to preserve the interpretation of the extended language $\mathcal{L}_0^{\mathbf{PAL}}$:

Proposition 5.1: F further preserves the interpretation of formulas in $\mathcal{L}_0^{\mathbf{PAL}}$ if it preserves the initial lifts of any injections, i.e. for any inclusion map $i : S \hookrightarrow X$, and for any object A in the fibre $\mathcal{A}_X$, we have

$$Fi^*A = i^*FA.$$

The reverse direction holds when the modal functor on $\mathcal{B}$ is injective on objects.

Proof Obviously, we should prove by induction, and the only case we need to care about is the interpretation of the PAL dynamic operators. Suppose F preserves the interpretation of the two formulas Φ, Ψ. We first show that if it commutes with initial lifts along injections, it also preserves the interpretation of the formula $[!\Phi]\Psi$. Let S be the set $[\![\Phi]\!]^V_A$, where V is some interpretation function on X. By assumption,

$$[\![\Phi]\!]^V_V = S = [\![\Phi]\!]^V_{FA}.$$

Thus, the restricted domain S and the induced inclusion map $i : S \hookrightarrow X$ is the same in both $\mathcal{A}$ and $\mathcal{B}$. By definition, we have

$$[\![[!\Phi]\Psi]\!]^V_A = \forall_i [\![\Psi]\!]^{i^*V}_{i^*A} = \forall_i [\![\Psi]\!]^{i^*V}_{Fi^*A} = \forall_i [\![\Psi]\!]^{i^*V}_{i^*FA} = [\![[!\Phi]\Psi]\!]^V_{FA}.$$

The first and last equality hold by definition of the interpretation of PAL update operators; the second holds by induction hypothesis, and the third holds because F commutes with initial lifts of injections. Thus, F preserves the interpretation of $\mathcal{L}_0^{\mathbf{PAL}}$.

On the other hand, suppose the semantic functor $(-)^+_{\mathcal{B}}$ is injective on object, and suppose for some object A in the fibre $\mathcal{A}_X$, and for some injection $i : S \hookrightarrow X$, we have i^*FA is *not* equal to Fi^*A. This in particular suggests that the associated operators $(i^*FA)^+_{\mathcal{B}}$, which we denote as m, and $(Fi^*A)^+_{\mathcal{B}}$, which we denote as m', on the power set $\wp(S)$ of the S are not identical, and they must disagree at some subset T of S. Then let V be an interpretation on X such that $V(p) = S$ and $V(q) = T$. Consider the interpretation of the formula $\langle !p \rangle \Box q$. On one hand, we have

$$[\![\langle !p \rangle \Box q]\!]^V_A = \exists_i [\![\Box q]\!]^{i^*V}_{i^*A} = \exists_i [\![\Box q]\!]^{i^*V}_{Fi^*A} = m'(T).$$

The second equality holds due to the fact that F is a modal functor, which preserves the interpretation of the basic modal language $\mathcal{L}$; the final equality holds because $T \subseteq S$ implies that $i^*V(q) = V(q) = T$. On the other hand, obviously we have

$$[\![\langle !p\rangle \Box q]\!]^V_{FA} = \exists_i [\![\Box q]\!]^{i^*V}_{i^*FA} = m(T).$$

By assumption, $m(T)$ does not coincide with $m'(T)$, and thus F does not preserve the interpretation of $\mathcal{L}_0^{\mathbf{PAL}}$ by definition. ∎

Remark 5.1: It can also be seen at this point that, dynamic extensions have a certain relative flavour. According to previous description, Definition 5.1 actually suggests that for any modal category, if it supports the interpretation of a particular fragment of modal logic $\mathcal{L}_0$, then it can also interpret the extended language $\mathcal{L}_0^{\mathbf{PAL}}$. The reverse is trivially true, because $\mathcal{L}_0^{\mathbf{PAL}}$ extends $\mathcal{L}_0$. Proposition 5.1 also suggests that, the preservation of the extended fragment of PAL dynamic logic requires *additional*, but *independent*, properties of the vertical transformation functor. Informally, this shows that model change, or at least definable model change, happens at another dimension in terms of the topology categorical structure that are used to interpret logical formulas. This is uniform theme of logical dynamics, shared by all the other types of dynamics that we are going investigate in this thesis. ◀

Using the above criterion, we can actually show that all the modal embeddings that we have described in Section 4.4 preserve the interpretation of certain fragments of modal languages extended with PAL updates:

Corollary 5.1: All the modal embedding appearing in the skeleton of the information landscape preserves the interpretation of the dynamic language $\mathcal{L}^{\mathbf{PAL}}$, or in fact the richer language $\mathcal{L}_\Sigma^{D,\mathbf{PAL}}$ extended with dependence atoms and multi-agent modalities.

Proof We have shown in Section 4.4 that all the embeddings appearing in the skeleton of the landscape are modal embeddings, thus preserves the interpretation of the language $\mathcal{L}_\Sigma^D$. By Proposition 5.1, we only need to verify that they also commutes with initial lifts of injections. From Example 5.1 and Example 5.3, we already know that the embeddings **Eqv** ↪ **Pre** ↪ **Kr** and **LMon** ↪ **Mon** ↪ **Nb** commutes with initial lifts of injections. The remaining case to check are the embeddings **Pre** ↪ **Top**, **Top** ↪ **LMon** and **Kr** ↪ **LMon**. As we have described in Chapter 4, the modal embedding **Top** ↪ **LMon** is a right adjoint, and thus commutes with every initial lift of single structured sources. Hence, we only need to consider the remaining two cases.

For the embedding **Pre** $\hookrightarrow$ **Top**, suppose we have a preorder $(X, \leq)$ and let $i : S \hookrightarrow X$ be an inclusion map. Let $\tau_{\leq}$ be the Alexandroff topology on X induced by $\leq$. Then the initial lift $i^*\tau_{\leq}$ calculated in **Top**, as Example 5.2 indicated, is given as follows,

$$i^*\tau_{\leq} = \{ S \cap U \mid U \in \tau_{\leq} \}.$$

Obviously, for any downward closed set U in $(X, \leq)$, the intersection $U \cap S$ is also downward closed in the restricted preorder $(S, \leq)$. On the other hand, it is easy to see that any downward closed set V in $(S, \leq)$ is equal to the intersection $V = S \cap V^*$, where V^* is upward closure of V in $(X, \leq)$. This means that the initial lift $i^*\tau_{\leq}$ calculated in **Top** coincide with the associated Alexandroff topology generated by $(S, \leq)$, hence the modal embedding **Pre** $\hookrightarrow$ **Top** commutes with initial lifts of injections.

For the embedding **Kr** $\hookrightarrow$ **LMon**, let (X, R) be any Kripke frame. Recall from Example 4.3 that the associated LM-neighbourhood frame E_R on X is given as follows: For any $x \in X$ and any $T \subseteq X$, we have

$$xE_RT \Leftrightarrow R[x] \subseteq T.$$

Now by Example 5.3, the initial lift along an inclusion map $i : S \hookrightarrow X$ is given as follows: For any $x \in S$ and $T \subseteq S$,

$$\begin{aligned} xi^*E_RT &\Leftrightarrow \exists T' \subseteq X[T = T' \cap S \,\&\, xE_RT'] \\ &\Leftrightarrow \exists T' \subseteq X[T = T' \cap S \,\&\, R[x] \subseteq T'] \\ &\Leftrightarrow R[x] \cap S \subseteq T \\ &\Leftrightarrow xE_{i^*R}T \end{aligned}$$

The second to least equivalence can be shown as follows: On one hand, if there exists $T' \subseteq X$ that $T = T' \cap S$ and $R[x] \subseteq T'$, then taking the intersection with S on both sides we get $R[x] \cap S \subseteq T' \cap S = T$. On the other hand, if $R[x] \cap S \subseteq T$, then consider the set $R[x] \cup T$. Obviously, we have

$$(R[x] \cup T) \cap S = (R[x] \cap S) \cup (T \cap S) = (R[x] \cap S) \cup T = T.$$

This indeed shows that the second equivalence hold, and thus we have again shown that the initial lift along injections commutes with the embedding **Kr** $\hookrightarrow$ **LMon**. ∎

Finally, we briefly indicate that Proposition 5.1 we have obtained above could not only be used to study the modal embeddings that appear in the skeleton of the landscape, we could also use it for all the reflections and coreflections. As we have discussed in Section 4.4 and Section 4.5, though they will not be modal functors for the canonical choice of semantic functors on the exemplar topological categories, for any reflection and coreflection we could change the semantic functor on the codomain category, so that the reflection and coreflection would indeed be a modal functor. Thus, if the reflection and coreflection preserves initial lifts of injections — which should be immediate for coreflections because it is by definition a concrete right adjoint — then they would also preserves the interpretation of the associated PAL extension of dynamic logic $\mathcal{L}^{\mathbf{PAL}}$, $\mathcal{L}_{\Sigma}^{\mathbf{PAL}}$ or $\mathcal{L}_{\Sigma}^{D,\mathbf{PAL}}$, though the interpretation of the modalities may not be the standard one for some topological categories. Here we do not record any further results on the preservation of initial lifts of injections by reflections and coreflections; we leave this for interested readers to find out.

5.2 Product Type Update

In this section, we are going to look at a more complicated type of update, which we call *product type update*. Simply from the name, it is easy to guess that the product type update is related to, and in fact generalises, the usual product update by an event model in dynamic epistemic logic, in our general context of topological categories.

However, our approach toward product type update —– one might say it is a bit unusual —– first deals with what we call the *empty product type update*, which could be viewed as certain degenerate case of product type update we are going to discuss later. The syntax and semantics of empty product update is very simple. Suppose we work within a modal category $\mathcal{A}$ that supports the interpretation of the fragment $\mathcal{L}_0$ of modal logic which extends $\mathcal{L}$. Then the following defines the basics of empty product type update:

Definition 5.2 (Empty Product Type Update): The syntax of $\mathcal{L}_0^{\mathbf{PRO}_0}$, of $\mathcal{L}_0$ extended with the empty product type update, is obtained by also allowing the formation of dynamic formulas of the form $\mathsf{U}\Phi$, where Φ belongs to $\mathcal{L}_0^{\mathbf{PRO}_0}$. For any model A in $\mathcal{A}$ over the set X, any evaluation function V on X, and any formula Φ in $\mathcal{L}_0^{\mathbf{PRO}_0}$, recursively we define the interpretation of $\mathsf{U}\Phi$ as follows,

$$[\![\mathsf{U}\Phi]\!]_A^V := [\![\Phi]\!]_{\top_X}^V,$$

where $\top_X$ is the top element in the fibre $\mathcal{A}_X$, viz. the indiscrete structure on X.

For most of the concrete examples of modal categories we have in mind, the empty product type operator U will actually be the *universal modality*. One can check this for the various relational structures, the topological spaces, etc.. The reason we treat it as *dynamics* is because, according to the above definition, it involves evaluating the formula in a *changed* model; just that in this case, the dynamics is pretty simple, because it is constant for any object in the same fibre. The following result on the preservation of interpretation of the extended language $\mathcal{L}_0^{\mathbf{PRO}_0}$ is also straight forward to see:

Proposition 5.2: Suppose both $\mathcal{A}$, $\mathcal{B}$ supports the interpretation of $\mathcal{L}_0$, and let $F : \mathcal{A} \to \mathcal{B}$ be a concrete functor that preserves the interpretation of formulas in $\mathcal{L}_0$. Then it further preserves the language $\mathcal{L}_0^{\mathbf{PRO}_0}$ if F preserves the top element in each fibre. The reverse hold when the semantic functor on $\mathcal{B}$ is injective on objects.

Proof It should be easy to see this for one self, essentially use the same technique of how we have proved Proposition 5.1. We leave this for the readers to check. ∎

Now let's get into the real game. The two types of dynamics we have introduced in the previous two sections, the PAL style dynamics and update by contraction, could be viewed as those types of updates that are induced purely by carving out or identifying some points in the original model. However, as we have discussed at the start of this chapter, we would also like to talk about other types of changes of the underlying set. Furthermore, the fact that we have equipped the updated model with the initial or final lifts as for the additional structure on it simply means that, the incoming information does not do anything more than what's necessary for the updated model. We would also like to devise an update procedure that could actively allow new information to come that changes the additional structure on the underlying set of states. As we will see, the product type update that will be introduced in this section partly fulfill both of these goals.

Let now $\mathcal{A}$ be a modal category we would like to work with, and suppose it supports the interpretation of a certain fragment of modal logic $\mathcal{L}_0$ that extends the basic modal logic $\mathcal{L}$. To be more specific, product type update will first allow us to make the state space more fine-grade, i.e. the previous single point in the set of states now could become several states in the updated model. However, we could not simply add arbitrary points to a model, since it cannot not be uniformly described by certain syntactic data. As we will see, this procedure is always done by referring to an additional parametrised set E, which could be thought of as some universe of more refined collection of states that we would like to consider about. Furthermore, it comes with an additional indexed family of propositions $\{\psi_e\}_{e \in E}$, where each formula ψ_e is in the language $\mathcal{L}_0$.① Intuitively, we could think of this family as specifying an identification procedure: For any original model A over the set X, the point e is considered as a more refined point of some state $x \in X$ iff x satisfies the formula ψ_e.

Additionally, we also allow the collection of more refined states E to come equipped with additional structure, which is described by some object B in the fibre $\mathcal{A}_E$ and a distinctive interpretation function W on E. These additional structures would then affect the original structure A and the evaluation V over the set X. The final updated model is then constructed through combining all these data in a natural and coherent way.

According to the above discussion, we first define below what a product type is. The readers may already see the familiarity between a product type and an event model in dynamic epistemic logic.②

Definition 5.3 (Product Type): A *product type* for the modal category $\mathcal{A}$ and the modal language $\mathcal{L}_0$ is a structure E, which is a tuple of the following type:

$$\mathsf{E} := \langle E, B, W, \{\psi_e\}_{e \in E} \rangle.$$

Above, E is a set, and B is an object in the fibre $\mathcal{A}_E$ over that set E, and W is an interpretation function on E. The family $\{\psi_e\}_{e \in E}$ is an E-indexed family of formulas within the language $\mathcal{L}_0$.

Notice that, strictly speaking, there is no need to mention the set E explicitly in the above definition, since given any object B in $\mathcal{A}$ we could get its underlying set by applying the forgetful functor that is part of the definition of a topological category. However, we find it more clear to explicitly mention the underlying set, since the family

① In fact, ψ_e could actually be formulas within the extended language $\mathcal{L}_0^{\mathbf{PRO}}$, which is the language $\mathcal{L}_0$ extended with product type update operators. We will not worry about this point too much.

② Though in an event model, no additional evaluation function W is required; this could be treated as a special case, as we will see later.

of formulas $\{\psi_e\}_{e\in E}$ is actually indexed by elements in E.

The product type update is then parameterised by a product type E. For any model A in $\mathcal{A}$ over the set X with a chosen evaluation function V on X, we define the updated model $\mathsf{E} \otimes_V A$ by a product type E according to the intuitive understanding of product type update discussed before. Its underlying set $\left|\mathsf{E} \otimes_V A\right|$, which we also denote as $E \otimes_V X$, should be given by the dependent sum as below,

$$E \otimes_V X := \sum_{e\in E} [\![\psi_e]\!]_A^V.$$

Since we have said that the indexed family of formulas $\{\psi_e\}_{e\in E}$ specifies a selection procedure, for any element e in E, it could be considered as a refined part of some original state $x \in E$, only if x satisfies ψ_e. Thus, the updated total space of states is naturally described by the above dependent sum.

Perhaps one can see this more clearly, by observing that there are two natural projection maps from $E \otimes_V X$ into E and X,

$$\begin{array}{ccc} E \otimes_V X & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_E}\big\downarrow & & \\ E & & \end{array}$$

The projection π_X then explicitly tells us which states in the updated underlying set $E \otimes_V X$ are more refined parts of the original state $x \in X$, given by its inverse image $\pi_X^{-1}(x)$.

The next part is to specify the additional structure on the underlying set $E \otimes_V X$, which is given by some object in the topological category $\mathcal{A}$ over that set. This is where the additional structure B over E takes effect. We define the model $\mathsf{E} \otimes_V A$ as the meet of the initial lifts of A and B along the above defined two projections,

$$E \otimes_V A := \pi_X^* A \wedge \pi_E^* B.$$

Similarly, we assign the updated evaluation function, which we denote as $W \otimes V$, on the updated set of states $E \otimes_V X$ is given as follows,

$$W \otimes V := \pi_X^* V \wedge \pi_E^* W.$$

Intuitively, this means that the more refined description of a state should combine the evaluation both coming from the original set X and the updated information.

In the usual context of product update in dynamic epistemic logic, the updated evaluation function is always determined by the one on X, which is simply given by the initial lift $\pi_X^* V$, without referring to any additional data on E. This could be viewed in our case by choosing the indiscrete evaluation function $\top_E$ on E, which interprets any propositional variable as the total set. Then $\pi_E^* \top_E$ will also be the indiscrete evaluation function on $E \otimes_V X$, since initial lifts preserves top elements. Taking intersection with it then amounts to nothing.①

Again there is the problem of transferring the evaluation back into the original set X. However in this case, the situation is more complicated than in PAL style dynamics or in update by contraction. This is because, in those two situations, the updated set is purely constructed using the information of the original set of states, while in product type update, the newly constructed model is defined by an additional parametrised set E. Hence, it makes sense if we can transport back into X by specifying an additional subspace S of E, considered as a special *detection* into E. With these data, we may then properly define the product type dynamics, which we also include the previously defined empty product type update, as follows:

Definition 5.4 (Product Type Dynamics): Let the modal category $\mathcal{A}$ supports the interpretation of the modal language $\mathcal{L}_0$ that extends $\mathcal{L}$. We define the product type dynamics extension of $\mathcal{L}_0$, which we denote as $\mathcal{L}_0^{\mathbf{PRO}}$, to be the one that is closed under empty product type update, and that also allows forming formulas of the form $[\mathsf{E}, S]\Phi$. Here, E is a product type for $\mathcal{A}$ and $\mathcal{L}_0$, and S is a subset of the underlying set E of the product type. Similarly, we also define its dual operator as follows,

$$\langle \mathsf{E}, S\rangle\Phi :\equiv \neg[\mathsf{E}, S]\neg\Phi.$$

Again to define the interpretation of formulas in $\mathcal{L}_0^{\mathbf{PRO}}$ in the modal category $\mathcal{A}$, we only need to specify the truth clause for the product type dynamics. For any model A in $\mathcal{A}$ over the set X, and any interpretation V on X, we define the interpretation of the formula $[\mathsf{E}, S]\Phi$ and $\langle \mathsf{E}, S\rangle\Phi$ as below,

$$[\![[\mathsf{E}, S]\Phi]\!]_A^V := \forall_{\pi_X}\left((S \otimes_V X) \to [\![\Phi]\!]_{E\otimes_V A}^{W\otimes V}\right),$$

$$[\![\langle \mathsf{E}, S\rangle\Phi]\!]_A^V := \exists_{\pi_X}\left((S \otimes_V X) \cap [\![\Phi]\!]_{E\otimes_V A}^{W\otimes V}\right).$$

① Another type of factual change deal within the context of dynamic epistemic logic is by giving *post-condition*, assigning the interpretation on the updated model based on whether the state in the old model satisfies certain proposition. This could also be treated categorically, though we will not discuss this in any detail here.

Here above, the set $S \otimes_V X$ is considered as a subset of $E \otimes_V X$, given by the following natural definition,

$$S \otimes_V X = \sum_{e \in S} [\![\psi_e]\!]_A^V.$$

The implication $\to$ and intersection $\cap$ is calculated in the power set $\wp(E \otimes_V X)$, considering it as a Boolean algebra. Again, there is a local version of truth condition associated to the definition above: For any $x \in X$, we have

$$A, x \vDash [\mathsf{E}, S]\Phi \Leftrightarrow \forall e \in S[A, x \vDash \psi_e \Rightarrow \mathsf{E} \otimes_V A, (e, x) \vDash \Phi],$$

$$A, x \vDash \langle \mathsf{E}, S \rangle \Phi \Leftrightarrow \exists e \in S[A, x \vDash \psi_e \,\&\, \mathsf{E} \otimes_V A, (e, x) \vDash \Phi].$$

Remark 5.2: Now we can say more about why we have called the previously defined dynamic modality U as the *empty* product type update, and include it in the final version of all product type update. The mathematical analogy is as follows: In any meet-semilattice, which means a partial order that has all finite meets, the empty meet is simply given by the top element of that lattice. This means that the top element is a certain degenerate form of meet, which is the categorical product in a lattice. Since the product type dynamics discussed above generally involves with taking binary meets in the fibre of a topological category, conceptually it makes sense to allow the empty product type update as a degenerate version of product type update. Thus, we will always look at them together. ◀

The above definition of the interpretation of formulas in the language $\mathcal{L}_0^{\mathbf{PRO}}$ could be even further generalised. To see this, we first need to reformulate Definition 5.4, especially for the last step of choosing a subset $S \subseteq E$ as a detection. The thing to notice is that, the implication and intersection taken with the set $S \otimes_V X$, has a categorical description as well. Let $i : S \hookrightarrow E$ be the inclusion, then we can view the newly defined subset $S \otimes_V X$ of $E \otimes_V X$ as the following *pullback*,

$$\begin{array}{ccc} S \otimes_V X & \overset{i}{\hookrightarrow} & E \otimes_V X \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi_E} \\ S & \underset{i}{\hookrightarrow} & E \end{array}$$

The induced map $j : S \otimes_V X \hookrightarrow E \otimes_V X$ is simply the inclusion map. Then notice that, for any subset T of $E \otimes_V X$, we actually have

$$(S \otimes_V X) \to T = \forall_j j \in (T), \quad (S \otimes_V X) \cap T = \exists_j j^{-1}(T).$$

This holds more generally for any inclusion map. Thus, using the above formulation, we could have an even more category theoretic description of the interpretation of formulas in $\mathcal{L}_0^{\mathbf{PRO}}$, as follows,

$$[\![[\mathsf{E}, S]\Phi]\!]_A^V := \forall_{\pi_X \circ j} j^{-1} [\![\Phi]\!]_{\mathsf{E} \otimes_V A}^{W \otimes V},$$

$$[\![\langle \mathsf{E}, S \rangle \Phi]\!]_A^V := \exists_{\pi_X \circ j} j^{-1} [\![\Phi]\!]_{\mathsf{E} \otimes_V A}^{W \otimes V}.$$

One thing we could do is to actually generalise the type of detection we would like to have, by allowing *arbitrary* functions $f : Y \to E$ mapping into E, and similarly take the pullback as below,

$$\begin{array}{ccc} Y \otimes_V X & \xrightarrow{g} & E \otimes_V X \\ {\scriptstyle \pi_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_E} \\ Y & \xrightarrow[f]{} & E \end{array}$$

In fact, in category theory, a function of the form $f : Y \to E$ is called a generalised element of E, which certainly matches our idea of *detection* considered in product type update. If we allow this more flexible type of detection, the product type update operator would now be of the form $[\mathsf{E}, f]$ and $\langle \mathsf{E}, f \rangle$, with $f : Y \to E$ being a generalised detection on E. And now, we could similarly define the interpretation of the dynamic language of product type update, allowing this extended types of detection, as below,

$$[\![[\mathsf{E}, f]\Phi]\!]_A^V := \forall_{\pi_X \circ g} g^{-1} [\![\Phi]\!]_{\mathsf{E} \otimes_V A}^{W \otimes V},$$

$$[\![\langle \mathsf{E}, f \rangle \Phi]\!]_A^V := \exists_{\pi_X \circ g} g^{-1} [\![\Phi]\!]_{\mathsf{E} \otimes_V A}^{W \otimes V},$$

where g is defined as the pullback of f along the projection π_E as above. We merely state the technical definition of this more general type of dynamics here; we leave the task of finding its potential application in dynamic logic to the readers.

Among all possible choice of detection, there are two types that are particularly important. One is a to look at a singleton set $\{e\}$ for some $e \in E$, and consider the inclusion $\{e\} \hookrightarrow E$. In this case, we also write $[\mathsf{E}, e]$ or $\langle \mathsf{E}, e \rangle$ instead of $[\mathsf{E}, \{e\}]$ or $\langle \mathsf{E}, \{e\} \rangle$. This is usually how it works in product update in dynamic epistemic logic, and it is easy to verity that the above defined semantic rule indeed recovers the usual truth condition of product update. Another one is to simply look at the identity function on E, which would take us stay within $E \otimes_V X$. In this case, we simply write $[\mathsf{E}]$ and $\langle \mathsf{E} \rangle$ without explicitly mentioning the associated subset.

Now is the time to look at how the general product type update would interact with vertical transformations. Suppose we already have a modal functor $F : \mathcal{A} \to \mathcal{B}$ that preserves the interpretation of the non-enriched language $\mathcal{L}_0$. The question is again that, what further properties of F should we require for it to preserve the extended language $\mathcal{L}_0^{\mathbf{PRO}}$.

This problem is, in some sense, trickier in the context of product type update than in the previous two types of dynamics, because it involves additional inputs consisting of a product type E and a detection $S \subseteq E$ — or more generally $f : Y \to E$. However, the definition of a product type E is relative to a specific modal category $\mathcal{A}$, because we have to choose an object B in the fibre $\mathcal{A}_E$ over its underlying set E. This way, the syntax of $\mathcal{L}_0^{\mathbf{PRO}}$ is, strictly speaking, dependent on the choice of the ambient modal category $\mathcal{A}$. However, any vertical transformation $F : \mathcal{A} \to \mathcal{B}$ obviously induces a mapping on product types in the two levels, respectively. We can use this to define the canonical translation between the language $\mathcal{L}_0^{\mathbf{PRO}}$ defined for two different modal categories $\mathcal{A}$ and $\mathcal{B}$, and use this instead to define what it means for a vertical connection F to preserve the interpretation of the language $\mathcal{L}_0^{\mathbf{PRO}}$:

Definition 5.5 (Preservation of the Dynamic Language Extended with Product Type Update): Suppose both the modal categories $\mathcal{A}$ and $\mathcal{B}$ supports the interpretation of the fragment of modal language $\mathcal{L}_0$. Let $F : \mathcal{A} \to \mathcal{B}$ be a concrete functor. For any product type $\mathsf{E} = \langle E, B, W, \{\psi_e\}_{e \in E} \rangle$ in $\mathcal{A}$, we define a product type $F\mathsf{E}$ in $\mathcal{B}$ as follows,

$$F\mathsf{E} := \langle E, FB, W, \{\psi_e\}_{e \in E} \rangle.$$

In other words, $F\mathsf{E}$ is the product type obtained by changing the object B in the fibre $\mathcal{A}_E$ to FB in $\mathcal{B}_E$. The *canonical translation* T between the two dynamic language $\mathcal{L}_0^{\mathbf{PRO}}$ associated to $\mathcal{A}$ and $\mathcal{B}$ is defined recursively, with the only non-trivial clause as follows,

$$T([\mathsf{E}, f]\Phi) \equiv [F\mathsf{E}, f]T(\Phi).$$

We then say that F preserves the interpretation of the language $\mathcal{L}_0^{\mathbf{PRO}}$ if the following holds: For any model A in $\mathcal{A}$ over the set X, any interpretation function V on X, and any formula Φ in $\mathcal{L}_0^{\mathbf{PRO}}$ associated to $\mathcal{A}$, we have

$$[\![\Phi]\!]_A^V = [\![T(\Phi)]\!]_{FA}^V.$$

Since now we have cleared the notion of what it means for a vertical transformation to preserve the extended dynamic logic of product type update, we can now look at when it does so.

Proposition 5.3: Let $F : \mathcal{A} \to \mathcal{B}$ be a modal functor that preserves the interpretation of the language $\mathcal{L}_0$. It further preserves the interpretation of the extended language $\mathcal{L}_0^{\mathbf{PRO}}$ —– in fact, with arbitrary detection into a product type —– if it preserves initial lifts of all single structured sources, and also finite meets fibre-wise. The reverse holds again when the semantic functor on $\mathcal{B}$ is injective on objects.

Proof Suppose F is such a concrete functor. We again show by induction that F preserves the interpretation of the extended language $\mathcal{L}_0^{\mathbf{PRO}}$, and by assumption and by Proposition 5.2, the only case we need to worry about is for the product type dynamic operator. Let A be a model in $\mathcal{A}$ over the set X, V be an interpretation function on X, and Φ be any formula in $\mathcal{L}_0^{\mathbf{PRO}}$ whose interpretation is preserved by F. Let $\mathsf{E} = \langle E, B, W, \{\psi_e\}_{e\in E}\rangle$ be an arbitrary product type, and $f : Y \to E$ be any detection into E. First of all, since F preserves the interpretation of $\mathcal{L}_0$, we have

$$(E \otimes_V X)_A = \sum_{e\in E} [\![\psi_e]\!]_A^V = \sum_{e\in E} [\![\psi_e]\!]_{FA}^V = (E \otimes_V X)_{\mathcal{B}}.$$

This means that the product type update, performed in $\mathcal{A}$ and $\mathcal{B}$ respectively, has the same underlying set, as well as the same projection maps π_E, π_X. Thus, we will simply denote it uniformly as $E \otimes_V X$. Now by definition, the additional structure of the updated model on it is calculated as follows,

$$E \otimes_V A = \pi_E^* B \wedge \pi_X^* A, \quad W \otimes V = \pi_E^* W \wedge \pi_X^* V.$$

Since F preserves finite meets fibre-wise and commutes with initial lifts of single structured source, it follows that

$$F(\mathsf{E} \otimes_V A) = F(\pi_E^* B \wedge \pi_X^* A) = \pi_E^* FB \wedge \pi_X^* FA = F\mathsf{E} \otimes_V FA.$$

Now by definition, we have

$$[\![[\mathsf{E}, f]\Phi]\!]_A^V = \forall_{\pi_X \circ g} g^{-1} [\![\Phi]\!]_{\mathsf{E}\otimes_V A}^{W\otimes V}$$

$$
\begin{aligned}
&= \forall_{\pi_X \circ g} g^{-1} [\![\Phi]\!]^{W \otimes V}_{F(\mathsf{E} \otimes_V A)} \\
&= \forall_{\pi_X \circ g} g^{-1} [\![\Phi]\!]^{W \otimes V}_{F\mathsf{E} \otimes_V FA} \\
&= [\![[F\mathsf{E}, f]\Phi]\!]^{V}_{FA}.
\end{aligned}
$$

In the above calculation, g is the pullback of f along π_E as we have defined before. The first and last equality holds simply by the definition of the interpretation of product type update; the second equality holds by induction hypothesis that F preserves the interpretation of Φ; the third holds by the previous proof that $F(\mathsf{E} \otimes_V A)$ coincide with $F\mathsf{E} \otimes_V FA$. Hence, this shows that F indeed preserves the interpretation of the extended dynamic language $\mathcal{L}_0^{\mathbf{PRO}}$.

For the reverse, we roughly need to show that any initial lift and any finite meets in the fibre could be represented by some product type update with a specific chosen product type. By Proposition 5.2, we may assume F is a modal functor that also preserves the top element in each fibre. We will first show that F preserves initial lifts of all single structured sources. Suppose for some function $\pi : E \to X$, F does not commute with the initial lifts on π in $\mathcal{A}$ and $\mathcal{B}$. This means that we have some object A in the fibre $\mathcal{A}_X$, such that $F\pi^* A$ and $\pi^* FA$ are two distinct objects in $\mathcal{B}_X$. Now since the semantic functor on $\mathcal{B}$ is injective on objects, the induced operators $(\pi^* FA)^+_{\mathcal{B}}$, which we denote as m, and $(F\pi^* A)^+_{\mathcal{B}}$, which we denote as m', on $\wp(X)$ would disagree on some subset T of E.

We then construct a product type E in $\mathcal{A}$, and choose an evaluation function V on X, such that the underlying set of the updated model and the two projection maps are as follows,

$$
\begin{array}{ccc}
E & \xrightarrow{\pi} & X \\
\| & & \\
E & &
\end{array}
$$

Consider the following product type E,

$$
\mathsf{E} = \langle E, \top^{\mathcal{A}}_E, W, \{p_e\}_{e \in E} \rangle,
$$

where the family of formulas is an E-indexed family of distinct propositional letters. For the evaluation function W on E, we require that for some propositional letter q distinct from p_e for any $e \in E$, we have $W(q) = T$. Now consider an evaluation function V on

X, such that for any $e \in E$ we have

$$V(p_e) = \{\pi(e)\},$$

which means that $[\![p_e]\!]^V_A$ is a singleton for any $e \in E$. We also requires that $V(q) = X$. Then by definition, we have

$$E \otimes_V X = \sum_{e \in E} [\![p_e]\!]^V_A = E,$$

and it is not hard to see that the projection map π_E is the identity on E, and π_X is simply given by π. Notice that, the above statement of the underlying set of the updated model remains true even if we have calculated it in $\mathcal{B}$.

The topology categorical structure is calculated as follows,

$$\mathsf{E} \otimes_V A = 1^*_E \top^{\mathcal{A}}_E \wedge \pi^* A = \pi^* A,$$

and for the induced product update in $\mathcal{B}$,

$$F\mathsf{E} \otimes_V FA = 1^*_E F \top^{\mathcal{A}}_E \wedge \pi^* FA = \top^{\mathcal{B}}_E \wedge \pi^* FA = \pi^* FA.$$

The above uses the fact that initial lifts preserves top elements, and the assumption that F preserves top elements in the fibre as well. As for the evaluation function $W \otimes V$, it is easy to calculate by definition that

$$(W \otimes V)(q) = W(q) \wedge \pi^{-1} V(q) = W(q) = T.$$

Finally, consider the interpretation of the formula $\langle \mathsf{E}, e \rangle \Box q$, where e is some element in E such that $e \in m(T)$ but $e \notin m'(T)$ (or vice versa). This amounts to choosing the detection as the inclusion of the single point $e : 1 \hookrightarrow E$, which results in the following pullback diagramme,

$$\begin{array}{ccccc} 1 & \overset{e}{\hookrightarrow} & E & \overset{\pi}{\longrightarrow} & X \\ \| & & \| & & \\ 1 & \overset{e}{\longrightarrow} & E & & \end{array}$$

Then by definition, we have the following calculation,

$$\begin{aligned} [\![\langle \mathsf{E}, e \rangle \Box q]\!]^V_A &= \exists_{\pi \circ e} e^{-1} [\![\Box q]\!]^{W \otimes V}_{\pi^* A} = \exists_{\pi \circ e} e^{-1} [\![\Box q]\!]^{W \otimes V}_{F\pi^* A} \\ &= \exists_{\pi \circ e} e^{-1} m'(T) = \varnothing. \end{aligned}$$

The first equality is due to the fact that $E \otimes_V A = \pi^* A$ as we have shown above; the second equality is by the fact that F preserves the interpretation of $\mathcal{L}_0$; and the final equality holds because we have assumed $e \notin m'(T)$, and e^{-1} has the effect of taking intersection with $\{e\}$. On the other hand, we have the other calculation as follows,

$$\begin{aligned}[\![\langle F\mathsf{E}, e\rangle \Box q]\!]^V_{FA} &= \exists_{\pi\circ e} e^{-1} [\![\Box q]\!]^{W\otimes V}_{F\mathsf{E}\otimes_V FA} = \exists_{\pi\circ e} e^{-1} [\![\Box q]\!]^{W\otimes V}_{\pi^* FA} \\ &= \exists_{\pi\circ e} e^{-1} m(T) = \{\pi(e)\}.\end{aligned}$$

These calculation are basically the same as before, only that in the final step, the result is a singleton $\{\pi(e)\}$ because $e \in m(T)$. This constructions shows that F would then not preserve the interpretation of the formula $\langle \mathsf{E}, e\rangle \Box q$ on this particular model. Hence, F must preserves the initial lift of any single structured sources.

Furthermore, we need to show that F preserves the binary meets fibre-wise as well. The basic idea is the same. Suppose F does not preserve binary meets in the fibre, then for some set X and some A, B in the fibre $\mathcal{A}_X$, we would have $F(A \wedge B)$ distinct from $FA \wedge FB$ in $\mathcal{B}_X$. Again, the operators m, m' associated to $F(A \wedge B)$ and $FA \wedge FB$ would differ on some subset T of X; we let $y \in X$ be the element in $m(T)$ but not $m'(T)$ (or vice versa).

We can then consider the product type X defined as follows,

$$\mathsf{X} = \langle X, B, \top^{\mathcal{V}}_X, \{p_x\}_{x\in X}\rangle.$$

We also consider the model A on X, with a chosen evaluation function V satisfying the following: For any $x \in X$, we have $V(p_x) = \{x\}$, and for another distinct variable q we have $V(q) = T$. The product type update would result in the following model,

$$X \otimes_V X = \sum_{x\in X} [\![p_x]\!]^V_A = X,$$

and the two projection maps are both the identity function 1_X on X. Again, this is independent of the modal categories $\mathcal{A}$ or $\mathcal{B}$. The topology categorical structure on the updated model, calculated in $\mathcal{A}$, is simply given as follows,

$$\mathsf{X} \otimes_V A = 1^*_X B \wedge 1^*_X A = A \wedge B.$$

In the modal category $\mathcal{B}$ however, we have

$$F\mathsf{X} \otimes_V FA = 1^* XFB \wedge 1^* XFA = FA \wedge FB.$$

In both cases, it is easy to see that the updated evaluation function $\top^{\mathcal{V}}_X \otimes V$ remains to be V itself.

By definition, consider the evaluation of the formula $\langle \mathsf{X}, y\rangle \Box q$. On one hand,

$$[\![\langle \mathsf{X}, y\rangle \Box q]\!]^V_A = \exists_y y^{-1}[\![\Box q]\!]^V_{A\wedge B} = \exists_y y^{-1}[\![\Box q]\!]^V_{F(A\wedge B)} = \exists_y y^{-1} m(T) = \{y\}.$$

On the other hand,

$$[\![\langle F\mathsf{X}, y\rangle \Box q]\!]^V_{FA} = \exists_y y^{-1}[\![\Box q]\!]^V_{FA\wedge FB} = \exists_y y^{-1} m'(T) = \varnothing.$$

Hence, this explicitly constructs a case where F does not preserve its interpretation. This completes the proof of this proposition. ∎

Corollary 5.2: For all the modal embeddings appearing in the skeleton of the landscape, only the ones among relational structures **Eqv** ↪ **Pre** and **Pre** ↪ **Kr**, and the one **Top** ↪ **LMon** preserves the interpretation of $\mathcal{L}^{\mathbf{PRO}}$, or in fact $\mathcal{L}^{D,\mathbf{PRO}}_{\Sigma}$.

Proof These mentioned embeddings preserve the interpretation of $\mathcal{L}^{\mathbf{PRO}}$ is because they are all right adjoint, thus preserves arbitrary meets in the fibre and all initial lifts of single structured sources. All other modal embeddings do not commute with all initial lifts of single structured sources, thus by Proposition 5.3, they do not preserve $\mathcal{L}^{\mathbf{PRO}}$. ∎

However, for those modal embeddings between other information levels, according to the description in Chapter 4, they all have concrete reflections, which is a concrete adjoint that commutes with all initial lifts of single structured sources and preserves arbitrary meets fibre-wise. Hence, it is again possible to change the semantic functor on the domain category of these reflections, and then these right adjoints would indeed preserve the interpretation of $\mathcal{L}^{\mathbf{PRO}}$, similar to the case of $\mathcal{L}^{\mathbf{CON}}$ we have discussed before.

Remark 5.3: Notice that, according to Proposition 5.3, for any modal functor F, preserving the interpretation of $\mathcal{L}^{\mathbf{PRO}}$ almost amounts to saying that F is a concrete right adjoint, but it is weaker in that it only requires F to preserve finite, not arbitrary, meets, fibre-wise. However, it is possible to further generalise product type update, to allow product type updated parametrised by not only one, but an arbitrary family $\{\mathsf{E}_i\}_{i\in I}$ of product types. However, this generalisation does not add anything new if we work within a single modal category. This is because, whenever we have a family $\{\mathsf{E}_i\}_{i\in I}$, we could always take their product $\prod_{i\in I}\mathsf{E}_i$ in a natural way, and the resulting product type update for the family $\{\mathsf{E}_i\}_{i\in I}$ must be equivalent to the result obtained by simply updating with $\prod_{i\in I}\mathsf{E}_i$, as one could easily see. However, the difference will emerge if we further look at vertical transformations. By naturally extending Definition 5.5, for any concrete functor $F : \mathcal{A} \to \mathcal{B}$, in the case of the update by a family of product types $\{\mathsf{E}_i\}_{i\in I}$ in $\mathcal{A}$, what we will consider then is the family $\{F\mathsf{E}_i\}_{i\in I}$ in $\mathcal{B}$, which is equivalent to the effect of $\prod_{i\in I} F\mathsf{E}_i$; however, for the single product type $\prod_{i\in I} \mathsf{E}_i$, it corresponds to the product type $F \prod_{i\in I} F\mathsf{E}_i$. At this point, it should be clear that for the functor F to preserve the interpretation of this generalised dynamic language of updating by family of product types, we need F to preserve arbitrary products in $\mathcal{A}$ as well as all initial lifts of single structured sources, which essentially amounts to saying that F is a concrete right adjoint. This is mathematically very interesting, because we have now provided a dynamic-logical characterisation of a functor being a concrete right adjoint between two topology categories. ◀

Remark 5.4: So far, we have completed the description of PAL style of update and product type in our general framework of topological categories. As could be seen in our development in the previous two sections, the fibre connections are indeed crucial for the interpretation of logical dynamics, and through Proposition 5.1 and Proposition 5.3, a precise connection have been established. However, in the whole chapter we have only looked at *pullback maps* between fibres, while having said nothing about *pushforwards*. This opens up the question of whether it is possible to introduce *new* types of dynamics that correspond to different types of pushforward maps.

For instance, we may intend to develop a dual variant of PAL style dynamics. Since PAL essentially relies on pullback maps along inclusions, the natural class of maps to look at for the dual dynamics are pushforward maps along *quotient maps*. In fact, in the logic dynamics literature, there are already several different attempts to define dynamics along quotient maps. Among them, one particular example is described in[61], where there the authors introduce dynamic operators induced by *filtrations* of models. And indeed, it is possible to use pushforwards to reformulate and generalise such types quotient dynamics in our framework. It still remains open whether we can formulate a dual version of product type update, and perhaps more importantly, even if we can, whether it is philosophically interesting to look at such dualisations. We leave these questions for future investigation. ◀

5.3 Examples of Product Type Update

All theoretical questions have already been answered for the product type update at this point. Let us now move to some concrete examples and special cases of product type update. In this section, we will see that many types of logical dynamics discussed in the literature could be recovered by our general notion of product type update, with specially chosen product types. For the first example, we will in fact show that PAL style updates can be viewed as special cases of product type update, which could be viewed as as a generalisation of the well-known fact that the usual public announcement logic is a fragment of the dynamic epistemic logic in Kripke models.

Example 5.5 (PAL Style Update as Product Type Update): It is well-known in the literature that PAL dynamics is a special case of product update in dynamic epistemic logic. Here in the context of horizontal dynamics in any topological categories, we further show that the generalised PAL style update could again be treated as special case of product type update we have defined above.

For any formula Φ, we construct a product type, which with an abuse of notation we also denote as $!\Phi$, as follows,

$$!\Phi := \langle 1, \top, \Phi \rangle.$$

In other words, $!\Phi$ is the product type with the underlying set being a singleton 1, equipped with the indiscrete structure in the fibre over 1, whose indexed family of formulas, which in this case only consists of one formulas, is given by Φ. For any model A in $\mathcal{A}$ over the set X, and any evaluation function V on X, by definition the updated model has the following underlying set,

$$1 \otimes_V X = \sum_{* \in 1} [\![\Phi]\!]^V_A = [\![\Phi]\!]^V_A.$$

The associated projection from $1 \otimes_V X$ into X now becomes the inclusion map $i : [\![\Phi]\!]^V_A \hookrightarrow X$, and the projection from $1 \otimes_V X$ into 1 is the uniquely determined. The topology categorical structure over this updated model is then given as below,

$$!\Phi \otimes_V A = \pi_1^* \top \wedge i^* A = i^* A.$$

This is because the initial lift π_1^* is a right adjoint, hence preserves meets, and in particular the empty meet $\top$. Similarly, the interpretation function on $!\Phi \otimes_V A$ is also given by

i^*V. This means that, the updated model $!\Phi \otimes_V A$ now becomes exactly the model we obtain after the PAL style update $!\Phi$.

Furthermore, the semantics of product type update also recovers the truth definition of PAL style update. Considering $!\Phi$ as a product type, according to Definition 5.4, we have

$$[\![[!\Phi]\Psi]\!]^V_A = \forall_i([\![\Phi]\!]^V_A \to [\![\Psi]\!]^{i^*V}_{i^*A}) = \forall_i [\![\Psi]\!]^{i^*V}_{i^*A},$$

$$[\![\langle !\Phi\rangle\Psi]\!]^V_A = \exists_i([\![\Phi]\!]^V_A \cap [\![\Psi]\!]^{i^*V}_{i^*A}) = \exists_i [\![\Psi]\!]^{i^*V}_{i^*A}.$$

In the above two lines of calculation, notice that the syntax suggests that we have chosen the whole space of the product type as detection, which in this case is actually the same as choosing the single object $*$ in 1 because it is a singleton. Both of the second equality holds because $[\![\Phi]\!]^V_A$ is the total set of the updated model $1 \otimes_V X$, and $1 \to a = a = 1 \wedge a$ in any Heyting algebra. This way, we have seen that the product type update recovers PAL style update, both syntactically and semantically, if we consider $!\Phi$ as a properly defined product type. ◀

Another important class of dynamics described in the literature of modal logic, especially in preference logic, includes various changes of the order relation over the underlying set of the model (cf.[17-18]). In the case of Kripke semantics of modal logic, many dynamic operators, including radical upgrade, suggestion, link deletion, etc., are of this form. The most general type of relational changes studied in the current modal logic literature is the programming style of dynamics treated in Propositional Dynamic Logic, or simply PDL. We will see how PDL could be generalised in our categorical setting later in this chapter. For now, we will first discuss the more general type of dynamics that involves changes of the additional information structure over a fixed underlying set of states in any modal category, that could already be treated in product type update introduced above:

Example 5.6 (Dynamics of Structural Change in Product Type Update): Let $\mathcal{A}$ be any modal category. Many of the dynamics of change of additional structures on a set of states are parametrised by a formula Φ. Let $2_{B,\Phi}$ be the following product type,

$$2_{B,\Phi} := \langle 2, B, \{0 : \neg\Phi, 1 : \Phi\}\rangle,$$

where $2 = \{0, 1\}$ is the set containing two distinct points, B is an object in the fibre $\mathcal{A}_2$, and the 2-indexed family of formulas is given by $\{\neg\Phi, \Phi\}$, where $\neg\Phi$ corresponds to 0 and Φ corresponds to 1. First notice that, for any model A in $\mathcal{A}$ over a set X, and for any interpretation function V on X, the underlying set of the updated model is X itself:

$$2 \otimes_V X = [\![\Phi]\!]_A^V \coprod [\![\neg\Phi]\!]_A^V = X.$$

And in fact, suppose $S = [\![\Phi]\!]_A^V$, the induced two projections from $2 \otimes_V X$ to 2 and X are given by the following diagramme,

$$\begin{array}{ccc} 2 \otimes_V X & = & X \\ {\scriptstyle \chi_S}\downarrow & & \\ 2 & & \end{array}$$

By definition, the projection onto X is the identity function itself, and the projection onto 2 is given by the characteristic function on S. This means that no matter what $\boldsymbol{B}$ and Φ we choose, the product type update over the product type $2_{B,\Phi}$ does not change the underlying set of the original model. However, the additional structure over the set $2 \otimes_V X$, which is X itself, indeed change. According to the object $\boldsymbol{B}$ we choose in the fibre $\mathcal{A}_2$, by definition we would have

$$2_{B,\Phi} \otimes_V A = \chi_S^* B \wedge A.$$

Then for any other formula Ψ, we can interpret the construct dynamic formula of product type update as follows,

$$[\![[2_{B,\Phi}\Psi]]\!]_A^V = [\![\Psi]\!]_{2_{B,\Phi} \otimes_V A}^V = [\![\langle 2_{B,\Phi}\Psi\rangle]\!]_A^V.$$

In other words, the two dual dynamic operators $[2_{B,\Phi}]$ and $\langle 2_{B,\Phi}\rangle$ now coincide, and both of them have the same effect of changing the topology categorical structure A over the underlying set X to the updated one $2_{B,\Phi} \otimes_V A$.

For example, interpreting the above construction in the familiar relational case **Pre** recovers many standard types of dynamical changes in the literature. In **Pre**, the fibre $\mathbf{Pre}_2$ has four elements: An element in $\mathbf{Pre}_2$ is either the discrete order $\perp$ or the indiscrete order $\top$ on 2, or the order where $\{0 < 1\}$, which we denote as **2**, or its reverse $\{1 < 0\}$. Usually, we will only look at the case of $\perp$ and **2**. The update by choosing $\top$ is trivial, since the top element in the fibre is preserves by any initial lifts, and taking meet with the top element amounts to nothing. Considering the order $\{1 < 0\}$ actually amounts to the same as considering the order $\{0 < 1\}$, but with a different choice of formula $\neg\Phi$

rather than Φ.

Now if we choose the discrete order $\perp$ on 2, for any characteristic function χ_S from X to 2, the order $\chi_S^*\perp$ on X is given as follows: For any $x, y \in X$, $x(\chi_S^*\perp)y$ iff x, y are both in S, or both in the complement $X\backslash S$. In other words, $\chi_S^*\perp$ is the equivalence relation on X, which has two distinct equivalence classes, one given by S and the other is its complement. Now for any original preorder $\leq$ on X, taking the intersection with $\chi_S^*\perp$ by definition will leave every ordering within S and the complement $X\backslash S$ untouched, but cutting out everything in between the two sets. This is exactly what the usual link-cutting operator $|\Phi$ does, i.e. for any Φ and Ψ, we have

$$[\![[2_{\perp,\Phi}\Psi]]\!]_A^V = [\![[|\Phi]\Psi]\!]_A^V.$$

This way, just as in the case of PAL style update, we can also view the link cutting operator $[|\Phi]$ as a special case of product type update, with the product type $|\Phi$ being $2_{\perp,\Phi}$.

On the other hand, if we have chosen the order $\mathbf{2}$, then the initial lift $\chi_S^*\mathbf{2}$ would be the order that relates everything within the two equivalence classes S and $X\backslash S$, as well as putting the class $X\backslash S$ entirely below S. In other words, for any $x, y \in X$, we have $x(\chi_S^*\mathbf{2})y$ if both x, y are in S or $X\backslash S$, or $x \in X\backslash S$ and $y \in S$. This way, for any preorder $\leq$ on X, taking the intersection with $\chi_S^*\mathbf{2}$ would have the same effect of cutting all the arrows from $\neg S$ to S, which is exactly what the suggestion dynamics $\#\neg\Phi$ does. In other words, for any Φ, Ψ we have

$$[\![[\mathbf{2}_\Phi]\Psi]\!]_A^V = [\![[\#\neg\Phi]\Psi]\!]_A^V.$$

The above discussion shows that, in the specific context of **Pre**, both the dynamics of link cutting and suggestion are special kinds of product type update we have described above. Of course, we could similarly interpret these constructions in other modal categories, like **LMon** or **Top**; we leave these possibilities for the interested readers. ◀

5.4 Horizontal and Vertical Interaction

Looking back at the informal information landscape presented in Figure 1.1, we have mentioned there that logical dynamics is essentially exploring the horizontal dimension of the landscape, by considering model change *within* a certain information level.

While on the other hand, the model transformations we have considered in Chapter 4, viz. the morphisms in $\mathfrak{Topc}$, acts in an inter-level way, filling the vertical dimension of the landscape. In a broad sense, both of these two types of operations could be viewed as *dynamical*, since they both trigger the change of one model to another. In fact, in a recent development, both of these two types of dynamics have been investigated from a modal logic perspective[62]. From this point of view, the natural question is to ask how the horizontal and vertical dynamics relate to each other.

In the logical literature, much work has been done towards such types of questions, in the name of *information tracking*, first introduced in the paper[24]; also see[11] and relevant chapters in[25]. The problem of tracking asks whether a horizontal dynamic operation on one information level could be *tracked* or not at another level, with respect to a specific model transformation acting as a vertical arrow. Formally, it is expressed in the form of *tracking diagrammes*. Suppose we have a modal category $\mathcal{A}$. For any model A in $\mathcal{A}$, a dynamic operator would always result in another model A'. Now if we have a certain vertical transformation F from $\mathcal{A}$ to $\mathcal{B}$, then both of the models A, A' would be mapped to FA, FA' in $\mathcal{B}$. The question now becomes whether we have a corresponding notion of dynamics in $\mathcal{B}$ making the below diagramme commute,

$$\begin{array}{ccc} A & \dashrightarrow & A' \\ \downarrow & & \downarrow \\ FA & \overset{?}{\dashrightarrow} & FA' \end{array}$$

The point is that, if for certain dynamic operation tracking is possible, then in this sense, such an operator has an exact counterpart at another level. There are also many variants related to the problem of tracking. For instance, we could also ask which types of vertical transformations are compatible for two fixed types of dynamics at two different levels.

We would like to argue that, our analysis in this chapter provides a slightly different angle on the problem of tracking. The first thing we would like to emphasis, though definitely well-known in the field of dynamic logic, is that a particular dynamic operator usually not only produces another model, but also certain *morphisms* between the updated model and the original one. In PAL style dynamics, this is the $\mathcal{A}$-morphism $i : \left|i^*A\right| \to |A|$ where $i : S \hookrightarrow X$ is the inclusion map induced by a public announcement; or in the product type update, we have the $\mathcal{A}$-morphism $\pi_X : \left|\mathsf{E} \otimes_V A\right| \to |A|$, with π_X the projection map from the updated product space $E \otimes_V X$ to X. As one can see from our development of logical dynamics in Section 5.1 and Section 5.2, such morphisms are crucial for our general definition of PAL style dynamics and product type update in an arbitrary topological category, and they contain more dynamic information than the mere two models alone, since they also records how they are related with each other.

Once we have these morphisms recording the dynamic information, the functoriality of a vertical transformation natually gives us a corresponding arrow in the codomain. For instance, in the case of PAL style update, the dynamic $\mathcal{A}$-morphism $f : i^*A \to A$ is mapped by F to the corresponding $\mathcal{B}$-morphism $Ff : Fi^*A \to FA$. In this case, there is always an arrow in the other information level that corresponds to the original dynamic morphism, resulting in a square of the following type,

$$\begin{array}{ccc} i^*A & \xrightarrow{f} & A \\ \downarrow & & \downarrow \\ Fi^*A & \xrightarrow[Ff]{} & FA \end{array}$$

However, this does not mean that, in our categorical perspective, all types of dynamics are tracked by an arbitrary vertical functor. The problem is that the resulting horizontal arrow Ff in the modal category $\mathcal{B}$ may *not* be the type of dynamics we want, or any types of dynamics at all. Again in the case of PAL style dynamics, our approach is to *compare* the resulting arrow Ff obtained by mapping the dynamic arrow f in $\mathcal{A}$ along F, and the naturally induced dynamic arrow $g : i^*FA \to FA$ in $\mathcal{B}$. Hence, our perspective is more align to the variant of the problem of tracking where we fix two types of dynamics at two different levels and ask whether a particular vertical transformation is compatible to them or not.

Now at least in the case of PAL style dynamics and product type dynamics, our categorical formulation has provided us very general tools from category theory to decide whether a particular concrete functor between two topological categories makes these two types of dynamics coincide in the two levels or not. For instance, if it has a concrete left adjoint, then general category theory implies that it preserves arbitrary fibre-wise meets and commutes with all pullback maps between fibres. Our categorical formulation of PAL style dynamics and product type updates more or less makes it evident that any such functor is compatible with these two types of updates, and this is why we can so effortlessly verify the results stated in Corollary 5.1 and Corollary 5.2, determining which model embeddings are compatible with PAL style updates or product type updates. In fact when they do coincide, our results stated in Proposition 5.1 and Proposition 5.3 are even slightly stronger than the mere existence of a tracking diagramme, in that they have also proved the interpretation of the corresponding fragments of dynamic logic remains *unchanged* under the vertical model transformation.

Of course, currently in this thesis we have only treated the case where we have the same type of dynamics in the two information levels. To investigate the more general type of tracking problem, where we look at whether a vertical transformation is compatible with two arbitrary types of dynamics in the two levels, requires us to take syntactic translations also into account, following Definition 4.7. We leave this for future works.

FIGURES

REFERENCES

[1] VON WRIGHT G H. An essay in modal logic[J]. British Journal for the Philosophy of Science, 1954, 5(18).

[2] HINTIKKA J. Knowledge and belief: An introduction to the logic of the two notions[J]. Studia Logica, 1962, 16.

[3] STALNAKER R C. A theory of conditionals[M]//RESCHER N. Studies in Logical Theory. Oxford: Blackwell, 1968: 98-112.

[4] LEWIS D. Counterfactuals[M]. Oxford: Basil Blackwell, 1973.

[5] VAN BENTHEM J, DUQUE D F, PACUIT E. Evidence logic: A new look at neighborhood structures[Z]. 2012.

[6] BALTAG A, VAN BENTHEM J. Knowability and continuity: a topological account of epistemic dependence[Z]. 2021.

[7] BALTAG A, VAN BENTHEM J. A simple logic of functional dependence[J]. Journal of Philosophical Logic, 2021: 1-67.

[8] VAN BENTHEM J. Modal Logic for Open Minds[M]. Stanford: Center for the Study of Language and Information, 2010.

[9] VAN BENTHEM J. Logical dynamics of information and interaction[M]. Cambridge University Press, 2011.

[10] ADRIAANS P, VAN BENTHEM J. Handbook of Philosophy of Information[M]. Elsevier, 2008.

[11] VAN BENTHEM J. The landscape of information in a logical perspective[J]. Unpublished note for the workshop "Logic in Theoretical Philosophy", Tsinghua University, 2018.

[12] PLAZA J. Logics of public announcements[C]//Proceedings 4th International Symposium on Methodologies for Intelligent Systems. 1989: 201-216.

[13] PLAZA J. Logics of public communications[J]. Synthese, 2007, 158(2): 165-179.

[14] BALTAG A, SMETS S. A qualitative theory of dynamic interactive belief revision[J]. Logic and the foundations of game and decision theory (LOFT 7), 2008, 3: 9-58.

[15] VAN BENTHEM J. Dynamic logic for belief revision[J]. Journal of applied non-classical logics, 2007, 17(2): 129-155.

[16] VAN DITMARSCH H P. Prolegomena to dynamic logic for belief revision[J]. Synthese, 2005, 147(2): 229-275.

[17] LIU F. Changing for the better: Preference dynamics and agent diversity[M]. ILLC, 2007.

[18] LIU F. Reasoning about preference dynamics: volume 354[M]. Springer Science & Business Media, 2011.

[19] VAN BENTHEM J, SMETS S. Dynamic logics of belief change[J]. ILLC preprints, 2015.

[20] VAN BENTHEM J, PACUIT E. Dynamic logics of evidence-based beliefs[J]. Studia Logica, 2011, 99(1): 61-92.

[21] EILENBERG S, MACLANE S. General theory of natural equivalences[J]. Transactions of the American Mathematical Society, 1945, 58(2): 231-294.

[22] AWODEY S. Structure in mathematics and logic: A categorical perspective[J]. Philosophia Mathematica, 1996, 4(3): 209-237.

[23] AWODEY S. Structuralism, invariance, and univalence[J]. Philosophia Mathematica, 2014, 22(1): 1-11.

[24] VAN BENTHEM J. Tracking information[M]//J. Michael Dunn on Information Based Logics. Springer, 2016: 363-389.

[25] CINÀ G. Categories for the working modal logician[D]. 2017.

[26] MCKINSEY J C C, TARSKI A. The algebra of topology[J]. Annals of mathematics, 1944: 141-191.

[27] MAKKAI M. Stone duality for first order logic: volume 107[M]. Elsevier, 1982: 217-232.

[28] MAKKAI M, REYES G E. First order categorical logic: model-theoretical methods in the theory of topoi and related categories: volume 611[M]. Springer, 2006.

[29] JOHNSTONE P T, et al. Sketches of an elephant: A topos theory compendium: Volumn 1, 2 [M]. Oxford University Press, 2002.

[30] PITTS A M. Categorical logic[R]. Citeseer, 1995.

[31] HERMIDA C. A categorical outlook on relational modalities and simulations[J]. Information and Computation, 2011, 209(12): 1505-1517.

[32] KISHIDA K. Categories for dynamic epistemic logic[A]. 2017.

[33] AWODEY S. Category theory[M]. Oxford university press, 2010.

[34] MAC LANE S. Categories for the working mathematician: volume 5[M]. Springer Science & Business Media, 2013.

[35] BLACKBURN P, VAN BENTHEM J F, WOLTER F. Handbook of modal logic[M]. Elsevier, 2006.

[36] AWODEY S, FORSSELL H. First-order logical duality[J]. Annals of Pure and Applied Logic, 2013, 164(3): 319-348.

[37] AWODEY S. Sheaf representations and duality in logic[J]. Joachim Lambek: The Interplay of Mathematics, Logic, and Linguistics, 2021, 20: 39.

[38] ADÁMEK J, HERRLICH H, STRECKER G E. Abstract and concrete categories : the joy of cats[M]. New York: Wiley, 1990.

[39] GARNER R. Topological functors as total categories[J]. Theory and Applications of Categories, 2014, 29(15).

[40] PREUSS G. Foundations of topology: an approach to convenient topology[M]. Springer Science & Business Media, 2011.

[41] BÉNABOU J. Introduction to bicategories[C]//Reports of the midwest category seminar. Springer, 1967: 1-77.

[42] PACUIT E. Neighborhood semantics for modal logic[M]. Springer, 2017.

[43] PULTR A, TRNKOVÁ V. Combinatorial, algebraic and topological representations of groups, semigroups and categories: volume 1980[M]. Prague, 1980.

[44] HOFMANN D, SEAL G J, THOLEN W. Monoidal topology: A categorical approach to order, metric, and topology: volume 153[M]. Cambridge University Press, 2014.

[45] BLACKBURN P, RIJKE M D, VENEMA Y. Cambridge tracts in theoretical computer science: Modal logic[M/OL]. Cambridge University Press, 2001. DOI: 10.1017/CBO9781107050884.

[46] TARSKI A. Zur grundlegung der boole'schen algebra. i[J/OL]. Fundamenta Mathematicae, 1935, 24(1): 177-198. http://eudml.org/doc/212745.

[47] JOHNSTONE P T. Stone spaces: volume 3[M]. Cambridge university press, 1982.

[48] VAN BENTHEM J. Correspondence theory[M]//Handbook of philosophical logic. Springer, 1984: 167-247.

[49] PATTINSON D. Coalgebraic modal logic: Soundness, completeness and decidability of local consequence[J]. Theoretical Computer Science, 2003, 309(1-3): 177-193.

[50] BALTAG A. A logic for coalgebraic simulation[J]. Electronic Notes in Theoretical Computer Science, 2000, 33: 42-60.

[51] STEINER A K. The lattice of topologies: structure and complementation[J]. Transactions of the American Mathematical Society, 1966, 122(2): 379-398.

[52] KELLY G M, KELLY M. Basic concepts of enriched category theory: volume 64[M]. CUP Archive, 1982.

[53] JOYAL A, TIERNEY M. An extension of the galois theory of grothendieck: volume 309[M]. American Mathematical Soc., 1984.

[54] ROSENTHAL K I. The theory of quantaloids[M]. Chapman and Hall/CRC, 1996.

[55] GIRARD J Y. Linear logic[J]. Theoretical computer science, 1987, 50(1): 1-101.

[56] ABRAMSKY S, VICKERS S. Quantales, observational logic and process semantics[J]. Mathematical structures in computer science, 1993, 3(2): 161-227.

[57] STUBBE I. An introduction to quantaloid-enriched categories[J/OL]. Fuzzy Sets and Systems, 2014, 256: 95-116. https://www.sciencedirect.com/science/article/pii/S0165011413003527. DOI: https://doi.org/10.1016/j.fss.2013.08.009.

[58] STREET R. Limits indexed by category-valued 2-functors[J]. Journal of Pure and Applied Algebra, 1976, 8(2): 149-181.

[59] YE L. Towards a relational treating of language and logical systems[M]//Context, Conflict and Reasoning. Springer, 2020: 97-113.

[60] YE L. Syntax and consequence relations–a categorical perspective[A]. 2021.

[61] VAN BENTHEM J, BEZHANISHVILI N. Modern faces of filtration[Z]. 2019.

[62] WESTERS F M. Dynamic logics for model transformations[Z]. 2022.

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声　明

本人郑重声明：所呈交的学位论文，是本人在导师指导下，独立进行研究工作所取得的成果。尽我所知，除文中已经注明引用的内容外，本学位论文的研究成果不包含任何他人享有著作权的内容。对本论文所涉及的研究工作做出贡献的其他个人和集体，均已在文中以明确方式标明。

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Publication

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[2] Grabmayr, B., Halbach, V., Ye, L. (2021) Varieties of Self-Reference in Metamathematics. 50 pages, now under review in *Journal of Philosophical Logic*.

[3] Ye, L. (2021). Syntax and Consequence Relations — A Categorical Perspective. arXiv preprint: 2109.04291. 32 pages, now under review in *Studia Logica*.

[4] Ye, L. (2022). Unification of Modal Logic via Topological Categories. Submitted for the *2022 Applied Category Theory* conference.

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综合论文训练记录表

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指导教师所在单位：清华大学哲学系

开题记录表

论文题目	A STRUCTURAL STUDY OF EPISTEMOLOGY AND INFORMATION IN A LOGICAL PERSPECTIVE

主要内容以及进度安排

Expected Outcome	Expected Finishing Time
Describe in detail the topological category structures of all the individual levels in the landscape of information	2021.11.15
Find the modal structure in each topological category corresponding to all the different semantic models	2021.11.30
Using the found abstract structure to transfer new logical designs like that of LCD into other modal logics	2021.12.31
Study the vertical transformations between different levels	2022.01.31
Using categorical framework to study the structural properties of different logical dynamics and compare them	2022.02.28
Combine the results of vertical transformation and horizontal dynamics to answer the tracking problem	2022.03.31
Finish the first draft of the thesis	2022.04.30
Systematic revision of the first draft to make it into a coherent piece, and prepare for the thesis defence	2022.05.31

导师评语

叶凌远已经阅读了与拟研究的问题相关的文献，梳理出了需要研究的问题。研究方法也已经准备就绪，可以开题。论文拟研究知识与信息的结构性质，这是知识论和逻辑学领域非常重要的问题，具有重要的理论意义。

刘奋荣

选题成绩（百分制）		
	文献查阅、外文阅读	98
	选题符合专业培养目标，体现综合论文训练基本要求	
	题目难易度	
	选题理论意义或应用价值	

考核组意见

论文研究问题明确思路清晰，进度安排得当，相关文献梳理完整有条理。同意开题。

教学负责人（签字）：陈壁生　　　　2021 年 11 月 10 日

附选题报告

中期检查记录表

论文题目	A Structural Study of Information in a Logical Perspective 关于信息结构的逻辑研究
论文进展情况	根据开题前撰写的开题报告，本论文预计采用逻辑学的视角，借用范畴论的数学工具，研究不同信息模型背后所具有的共同抽象结构，以期更加深入地理解主体处理信息、根据信息进行推理的模式。 根据开题报告所设定的论文目标，本论文预计共分为五章，分别探讨本论文题目的研究背景及内容介绍、所选取的数学理论、信息模型背后的抽象逻辑结构、不同信息模型之间的关联、以及其与动态逻辑的关系。 目前，已完成前四章及部分第五章论文初稿的撰写，超出开题时所预设的实际，实际进展良好。后期需要尽快完成剩余部分的论文撰写，并根据导师及其他教授的反馈意见进行修改、补充。预计能够按照规定时间顺利完成整个论文项目。
中期成绩（百分制）	98
考核组意见	本文准备充分，进展良好，借用逻辑学视角对主体处理信息的模式进行研究。建议根据导师及其它老师建议进一步修改。

教学负责人（签字）：刘奋荣 陈壁生　　2022年 2月 24日

论文评阅记录表

论文题目	A Structural Study of Information in a Logical Perspective
指导教师评语	Ye Lingyuan's bachelor's thesis is a remarkably mature analysis of models and logics for information structures and their updates. It puts many existing systems in one mathematical setting, using a new category-theoretic perspective developed for this purpose. The 'topological categories' presented here suggest many new directions, including more systematic ways of designing and connecting logical systems for the varieties of update in the recent literature on inquiry, agency and games. But the category-theoretic content of the thesis is also substantial in its own right. This thesis is a clear case of 'thinking' about larger issues that goes beyond mere problem solving. **指导教师签字：** **2022 年 5 月 24 日**
评阅教师评语	作为一篇学士学位论文，该文在以下几方面都是极为出众的。（1）理论高度——文章没有自我局限于具体的细节问题，而是尝试在范畴论框架中，对常被用来为信息活动建模的几类数学模型进行概括和抽象。（2）主旨清晰——文章既没有浮于抽象，也没有刻意炫技；遵循技术工具为理论动机服务的原则，论文在缺乏既有工具的场合作出了方法意义上的创新。（3）文本写作质量——文章不仅拥有远超通常意义上学士学位论文的篇幅，而且具有出色的文本品质，符号讲究、排版干净美观。 综合来看，这是一篇全方位超出一般意义上学士学位论文水准的优秀论文。 **评阅教师签字：** **2022 年 06 月 09 日**

论文答辩记录表

<table>
<tr><td>论文题目</td><td colspan="2">A Structural Study of Information in a Logical Perspective</td></tr>
<tr><td rowspan="5">综合能力评价</td><td>查阅文献资料能力</td><td rowspan="5">100</td></tr>
<tr><td>综合运用知识能力</td></tr>
<tr><td>研究方案的设计能力</td></tr>
<tr><td>研究方法和手段的运用能力</td></tr>
<tr><td>外文应用能力</td></tr>
<tr><td rowspan="5">成果质量评价</td><td>文题相符</td><td rowspan="5">100</td></tr>
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<tr><td>写作规范</td></tr>
<tr><td>成果的理论或应用价值</td></tr>
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<tr><td>回答问题</td></tr>
<tr><td colspan="2">答辩成绩（以上三项的综合考查结果，百分制）</td><td>100</td></tr>
<tr><td>答辩小组评语</td><td colspan="2">叶凌远同学的论文采用范畴论研究信息结构和信息更新，提供了一个看待模态逻辑的新的统一视角。这个视角十分新颖，论文证明的结果具有原创性。论文用英文写作，思路清晰，符合写作规范。答辩时，该同学的语言流畅，反应迅速，能够清楚、准确地回答答辩委员所提出的问题。
综合指导教师、评阅人的意见和该同学在答辩过程中的表现，答辩组经过认真讨论，一致认为该同学的学位论文达到了本科论文的要求，同意通过该同学的毕业论文答辩。

组长签字：刘奋荣</td></tr>
</table>

总成绩（百分制）： **99**

说明：总成绩为开题成绩、中期成绩、答辩成绩三项成绩的加权平均，加权值参考《清华大学本科生综合论文训练管理条例》。

教学负责人（签字）： 陈壁生　　　　**2022 年 6 月 9 日**