SHEAF REPRESENTATION OF HEYTING ALGEBRAS

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This is a short note covering the sheaf representation of finitely presented Heyting algebras described in the book *Sheaves, Games, and Model Completions* by Ghilardi and Zawadowski.

This file is subject to change!.

1. Syntactic Category of Algebraic Theories

Let $\mathcal{H}$ be the *left exact* syntactic category of an algebraic theory $\mathbb{H}$. Its objects are conjunction of equations $E(\overline{x})$, and morphisms are indexed by terms

$$t_1(\overline{x}), \cdots, t_n(\overline{x}) : E(\overline{x}) \to F(\overline{y}),$$

where $|\overline{y}| = n$, such that

$$E(\overline{x}) \wedge \bigwedge_{i \leq n} y_i = t_i(\overline{x}) \vdash_{\mathbb{H}} F(\overline{y}).$$

Two such indexing $\{t_i(\overline{x})\}_{i \leq n}$ and $\{s_i(\overline{x})\}_{i \leq n}$ represent the same morphism iff

$$E(\overline{x}) \vdash_{\mathbb{H}} \bigwedge_{i \leq n} t_i(\overline{x}) = s_i(\overline{x}).$$

It is well-known that this defines a left exact category, dual to the category of finitely presented algebras

$$\mathcal{H} \simeq \mathbb{H}\text{-}\mathrm{Alg}_{\mathrm{fp}}^{\mathrm{op}}.$$

The equivalence takes the following form,

$$\begin{array}{ccc} E(\overline{x}) & \longmapsto & \mathcal{H}(\overline{x})/E(\overline{x}) \\ {\scriptstyle t_i(\overline{x})}\downarrow & & \uparrow{\scriptstyle y_i \mapsto t_i(\overline{x})} \\ F(\overline{y}) & \longmapsto & \mathcal{H}(\overline{y})/F(\overline{y}) \end{array}$$

which maps each $E(\overline{x})$ to the quotient of the free algebra $\mathcal{H}(\overline{x})$ on $|\overline{x}|$ many generators by the equations $E(\overline{x})$; and maps each morphism $t_i(\overline{x})$ to the corresponding morphism by mapping the generator $y_i \in \mathcal{H}(\overline{y})/F(\overline{y})$ to the element $t_i(\overline{x})$ in $\mathcal{H}(\overline{x})/E(\overline{x})$. These are all standard, and the upshot is that

$$\mathbf{Lex}(\mathcal{H}, \mathbf{Set}) \simeq \mathbf{Lex}(\mathbb{H}\text{-}\mathrm{Alg}_{\mathrm{fp}}^{\mathrm{op}}, \mathbf{Set}) \simeq \mathbf{Ind}(\mathbb{H}\text{-}\mathrm{Alg}_{\mathrm{fp}}) \simeq \mathbb{H}\text{-}\mathrm{Alg}.$$

A morphism $t_i(\overline{x}) : E(\overline{x}) \to F(\overline{y})$ is monic in $\mathcal{H}$, iff it is provably monic, viz. iff

$$E(\overline{x}) \wedge E(\overline{x}') \wedge \bigwedge_{i \leq n} t_i(\overline{x}) = t_i(\overline{x}') \vdash \overline{x} = \overline{x}'.$$

Such a morphism is *regular monic*, if in the dual category $\mathbb{H}\text{-}\mathrm{Alg}_{\mathrm{fp}}$, it is regular epic, or equivalently *surjective*, hence corresponds to certain diagramme as follows,

$$\begin{array}{ccc} \mathcal{H}(\overline{y})/F(\overline{y}) & \xrightarrow{t_i(\overline{x})} & \mathcal{H}(\overline{x})/E(\overline{x}) \\ & \searrow & \downarrow \cong \\ & & \mathcal{H}(\overline{y})/F(\overline{y}) \wedge G(\overline{y}) \end{array}$$

Hence, regular monos in $\mathcal{H}$ are isomorphic to one of the form $E(\overline{x}) \hookrightarrow F(\overline{x})$ where $E(\overline{x}) \vdash_{\mathbb{H}} F(\overline{x})$.

Recall that we say the theory $\mathbb{H}$ satisfies the *Beth-definability condition*, if for any formula $E(\overline{x}, y)$, if we have

$$E(\overline{x}, y) \wedge E(\overline{x}, z) \vdash_{\mathbb{H}} y = z,$$

then there exists a term $t(\overline{x})$ such that

$$E(\overline{x}, y) \vdash_{\mathbb{H}} y = t(\overline{x}).$$

Theorem 1.1. *In $\mathcal{H}$, every mono is regular iff $\mathbb{H}$ satisfies the Beth-definability condition.*

Proof. We only show the only if direction, leaving the other direction for exercise. Notice that given a formula $E(\overline{x}, y)$ satisfying

$$E(\overline{x}, y) \wedge E(\overline{x}, z) \vdash_{\mathbb{H}} y = z,$$

exactly means that the following projection in $\mathcal{H}$ is monic,

$$\overline{x} : E(\overline{x}, y) \to \top(\overline{x}).$$

By assumption, it is isomorphic to a regular mono as follows,

$$\begin{array}{ccc} E(\overline{x}, y) & \xrightarrow[\cong]{\overline{x}} & F(\overline{x}) \\ & \overline{x} \searrow & \downarrow \overline{x} \\ & & \top(\overline{x}) \end{array}$$

The inverse of $\overline{x} : E(\overline{x}, y) \to F(\overline{x})$ must be of the form

$$\overline{x}, t(\overline{x}) : F(\overline{x}) \to E(\overline{x}, y),$$

where $F(\overline{x}) \vdash E(\overline{x}, t(\overline{x}))$. This is the desired term we want. □

Example 1.2. Let $\mathbb{H}$ be the category theory of Heyting algebras. Since propositional intuitionistic logic has interpolation, $\mathbb{H}$ satisfies Beth-definability. This means that in $\mathcal{H}$, the syntactic category of Heyting algebras, all monos are regular. ◊

In this note we are particularly interested in $\mathcal{H}$ being the syntactic category of Heyting algebras. We would like to show that it is a Heyting category. One of the proof strategies is by embedding it into a sheaf topos

$$\mathcal{H} \hookrightarrow \mathrm{Sh}(\mathbf{Por}_{\mathrm{f}}, J),$$

and by showing that $\mathcal{H}$ is closed under the Heyting category operations which exist in $\mathrm{Sh}(\mathbf{Por}_\mathrm{f}, J)$. Here $\mathbf{Por}_\mathrm{f}$ is the category of finite posets and open maps between them. In particular, $\mathbf{Por}_\mathrm{f}$ is the dual category $\mathbf{HA}_\mathrm{f}$ of finite Heyting algebras, and this duality is exactly what allows us to represent general Heyting algebras. From now on we focus on the study of Heyting algebras, and our $\mathcal{H}$ and $\mathbb{H}$ will always denote for Heyting algebras.

2. The Category of Sheaves on Finite Kripke Frames

In this section we study the sheaf topos $\mathrm{Sh}(\mathbf{Por}_\mathrm{f}, J)$. As mentioned, the category $\mathbf{Por}_\mathrm{f}$ is the category of finite posets with open maps. A monotone map is open iff it takes upper closed subsets to upper closed subsets. In particular, open surjections between posets are exactly p-morphisms if we think of them as Kripke frames.

The topology J on $\mathbf{Por}_\mathrm{f}$ is exactly taking into account this bisimulation information: It assigns each $P \in \mathbf{Por}_\mathrm{f}$ the covering sieves which contain a jointly surjective family of maps $\{Q_i \to P\}$. To show this consists of a basis of a topology on $\mathbf{Por}_\mathrm{f}$, we first notice:

Lemma 2.1. *Open maps between posets are closed under composition and pullback in* $\mathbf{Pos}$.

Proof. Composition is evident. Say we have a open map $p : Q \to P$. Taking its pullback along any monotone map $f : R \to P$ results in

$$\begin{array}{ccc} R \times_P Q & \longrightarrow & Q \\ \downarrow & & \downarrow{\scriptstyle p} \\ R & \xrightarrow[f]{} & P \end{array}$$

Now suppose we have $(r, q) \in R \times_P Q$ and $r \leq r'$ in Q. By p being open, since $p(q) = f(r) \leq f(r')$, we can find $q' \in Q$ such that $p(q') = f(r')$. We have then located $(r, q) \leq (r', q')$ lying over $r \leq r'$. Hence the projection $R \times_P Q \to R$ is open. □

Remark 2.2. However, in general the pullback in $\mathbf{Por}_\mathrm{f}$ is not computed as in $\mathbf{Pos}_\mathrm{f}$, since otherwise pushouts in $\mathbf{HA}_\mathrm{f}$ will be computed the same as in $\mathbf{DL}_\mathrm{f}$, and this is just not true. But finite colimits in $\mathbf{Por}_\mathrm{f}$ is computed the same as in $\mathbf{Pos}_\mathrm{f}$. In our notes, pullbacks will always denote pullbacks taken in $\mathbf{Pos}$, unless otherwise stated. ◊

Lemma 2.3. *J is a Grothendieck topology on* $\mathbf{Por}_\mathrm{f}$.

Proof. Maximal sieves contain the identity, hence is a covering sieve. Suppose we have a covering sieve S on P containing the jointly surjective family $\{Q_i \to P\}$, then for any $f : R \to P$, the family pulled back along f, viz. $\{Q_i \times_P R \to R\}$, is again a jointly surjective family in $\mathbf{Por}_\mathrm{f}$ by Lemma 2.1. Finally, jointly surjective families are evidently composible. □

Inside the site $(\mathbf{Por}_\mathrm{f}, J)$ there is a dense subsite $(\mathbf{Por}_{\mathrm{f},*}, J)$, where $\mathbf{Por}_{\mathrm{f},*}$ is the full subcategory of *rooted* finite posets. A poset is rooted iff it has a least element.

Lemma 2.4. *Any finite poset P admits a jointly surjective family $\{P_i \to P\}$ in $\mathbf{Por}_{\mathrm{f}}$, where each $P_i \in \mathbf{Por}_{\mathrm{f},*}$.*

Proof. Since P is finite, it has finitely many minimal elements x_i, and we take P_i to be $\uparrow x_i$. Each P_i is evidently rooted, with the inclusion $P_i \to P$ an open map. Furthermore, since any element of P lies over some minimal element, the induced inclusion $\{P_i \to P\}$ is jointly surjective. □

Corollary 2.5. *There is an equivalence of categories*

$$\mathrm{Sh}(\mathbf{Por}_{\mathrm{f}}, J) \simeq \mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J).$$

Proof. By Lemma 2.4 and the fact that $\mathbf{Por}_{\mathrm{f},*}$ is a full subcategory of $\mathbf{Por}_{\mathrm{f}}$, $(\mathbf{Por}_{\mathrm{f},*}, J)$ is a dense subsite of $(\mathbf{Por}_{\mathrm{f}}, J)$, hence they induce the same sheaf topos. □

Remark 2.6. One nice thing about the presentation $(\mathbf{Por}_{\mathrm{f},*}, J)$ is that, the topology J on $\mathbf{Por}_{\mathrm{f},*}$ is *regular*: For any jointly surjective family $\{P_i \to P\}$ with P rooted, there must exists some $P_i \to P$ whose image contains the root; and by this map being open, it must itself be an open surjection. Hence, J is a regular topology on $\mathbf{Por}_{\mathrm{f},*}$, and the above equivalence shows that the topos $\mathrm{Sh}(\mathbf{Por}_{\mathrm{f}}, J)$ is a regular topos. ♢

Another characterisation of the topology J on $\mathbf{Por}_{\mathrm{f},*}$ is that it is the canonical topology:

Proposition 2.7. *J is the canonical topology on $\mathbf{Por}_{\mathrm{f},*}$, i.e. it is the finest topology such that all representable functors are sheaves.*

Proof. Take P to be any finite rooted poset. We show $\mathbf{Por}_{\mathrm{f},*}(-, P)$ is a sheaf for J. Suppose we have a covering sieve S on Q containing an open surjection $f : R \twoheadrightarrow Q$. Given a compatible family $\{x_g : R' \to P\}$ for this sieve, to extend to some $x : Q \to P$, we first take the pullback of f along itself,

$$\begin{array}{ccc} R \times_Q R & \xrightarrow{p_1} & R \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle f} \\ R & \xrightarrow[f]{} & Q \end{array}$$

Notice that $R \times_Q R$ is also rooted, because R, Q are and f is surjective. Furthermore, this will also be a pushout in $\mathbf{Pos}$ because f is a regular epi, hence in $\mathbf{Por}_{\mathrm{f},*}$ as well. By compatibility,

$$x_f p_0 = x_{f p_0} = x_{f p_1} = x_f p_1,$$

hence the universal property uniquely induces a map $x : Q \to P$. We need to show this x is compatible with other maps $g : R' \to Q$ in S. Given any such map, again we take the pullback of g along f,

$$\begin{array}{ccccc} R'' & \xrightarrow{i} & R' \times_Q R & \xrightarrow{p_1} & R \\ & {\scriptstyle h}\searrow & {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle f} \\ & & R' & \xrightarrow[g]{} & Q \end{array}$$

Notice that now $R' \times_Q R$ need not to be rooted, because g is not surjective. However, since p_0 is an open surjection, we can choose an element in the fibre product $R' \times_Q R$ that is mapped to the root, and take its upper set to be R'' as above. This way, R'' is again rooted, and the composition $R'' \twoheadrightarrow R'$ is still an open surjection. By compatibility again,

$$x_g h = x_{gh} = x_{fp_1 i} = x_f p_1 i = x f p_1 i = x g h,$$

which says that x_g and xg agrees when precomposed with h. But h is surjective, hence $x_g = xg$, which implies x is indeed compatible with all $g \in S$.

On the other hand, suppose we have any subcanonical topology K on $\mathbf{Por}_{\mathrm{f},*}$. Let S be a K-covering sieve on Q. We show that S must also be a covering sieve in J, i.e. it must contains a map $f : R \to Q$ which is surjective. If not, consider the map $q : Q_* \to Q$, where Q_* is obtained by universally adding a root to Q, and q maps the new root to the original root in Q. The pullback $q^*(S)$ will be a K-covering on Q_*. However, this family has two ways of extending to a map Q_* to itself, viz. either the identity or the composite $Q_* \twoheadrightarrow Q \hookrightarrow Q_*$. This proves $K \subseteq J$. □

The above proof only uses the pullback and pushout property of open surjections in $\mathbf{Pos}$, where the pushout actually coincides with that in $\mathbf{Por}_{\mathrm{f},*}$. Hence, the proof actually works for arbitrary poset L. In particular, any L gives us a sheaf $\underline{L}$ on $\mathbf{Por}_{\mathrm{f},*}$ by Yoneda,

$$\underline{L}(P) := \mathbf{Pos}(P, L).$$

Proposition 2.8. *There is a faithful embedding*

$$\underline{(-)} : \mathbf{Pos} \rightarrowtail \mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J)$$

which preserves all limits in $\mathbf{Pos}$.

Proof. As mentioned, the proof of Proposition 2.7 implies that for any poset L, the presheaf $\mathbf{Pos}(-, L)$ will be a sheaf for $(\mathbf{Por}_{\mathrm{f},*}, J)$. $\mathbf{Pos} \rightarrowtail \mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J)$ is faithful, because the difference of two maps $f, g : L \to M$ can be detected by some $x : 1 \to L$ with now 1 a rooted poset. Furthermore, for any limit of posets $\varprojlim_{i \in I} L_i$ and any rooted poset P, we have

$$\mathbf{Pos}(P, \varprojlim_{i \in I} L_i) \cong \varprojlim_{i \in I} \mathbf{Pos}(P, L_i).$$

Since limits of sheaves are computed point-wise, it follows the embedding preserves all limits. □

Remark 2.9. The above embedding is not full. The problem is that open maps are too restrictive to guarantee that all natural transformations between two representables comes from some *monotone* map between the original posets. Concretely, say we have a natural transformation

$$\eta : \mathbf{Pos}(-, L) \to \mathbf{Pos}(-, W),$$

which indeed gives us a map

$$\eta_1 : \mathbf{Pos}(1, L) \cong |L| \to \mathbf{Pos}(1, W) \cong |W|.$$

However, to show that this map is monotone, we need to look at the naturality condition w.r.t. the following squares,

$$\begin{array}{ccc} \mathbf{Pos}(1,L) & \xrightarrow{\eta_1} & \mathbf{Pos}(1,W) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle i^*} \\ \mathbf{Pos}(2,L) & \xrightarrow[\eta_2]{} & \mathbf{Pos}(2,W) \end{array}$$

In particular, $\mathbf{Pos}(2, L)$ classifies pairs (x, y) in L that $x \leq y$. In usual, where we can take i to be both 0 or 1, the above naturality will imply that $\eta_2(x, y) = (\eta_1 x, \eta_1 y)$, hence η_1 must be monotone. However, if we restricts to open maps, we could only take i to be 1, since 0 is *not* open. This then does not imply the monotonicity of η_1, and in general there could be more natural transformations. $\Diamond$

3. Games and Sheaves

In this section, we will show that the sheaf condition we put in the previous section on the site $(\mathbf{Por}_{\mathrm{f},*}, J)$ can be understood as closure under certain type of bisimulation, and the way to see this is to introduce a bisimulation game. Given two elements $u \in \underline{L}(P)$ and $v \in \underline{L}(Q)$ of the representable sheaf $\underline{L}$, we design a game $\mathsf{G}_\infty(u, v)$. There are two players A (Abelard) and E (Eloise), and at each round E first chooses an element in either P or Q, and then A chooses an element in the other poset, subject to the following conditions: Suppose the last chosen pair is $\langle p_n, q_n \rangle$, then for the next round $\langle p_{n+1}, q_{n+1} \rangle$ we must have

- $u(p_{n+1}) = v(q_{n+1})$;
- $p_n \leq p_{n+1}$ and $q_n \leq q_{n+1}$.

Furthermore, we declare the 0-th round of the game to start at $\langle p_0, q_0 \rangle = \langle *, * \rangle$, viz. the two roots of the posets. E wins the game if at some finite step A can only make an unvalid move, while if the game can be played infinitely, A wins the game. In particular, if $u(*) \neq v(*)$, A looses immediately. We also denote the game $\mathsf{G}_n(u, v)$ if we restrict the game to be played in maximal n-rounds, and declare A to win if she has successfully played n-rounds. We also denote a non-symmetric modification of the game $\mathsf{G}'_\infty(u, v)$ or $\mathsf{G}'_n(u, v)$ for $n > 0$, if we specify that E must choose an element of P in the *first* round.

Let $\mathbb{N} \cup \{\infty\}$ denote the poset of natural numbers with an infinite element added at top. We will use $\alpha \in \mathbb{N} \cup \{\infty\}$ as our index of the game we have defined above.

Definition 3.1. *For $u \in \underline{L}(P)$ and $v \in \underline{L}(Q)$, we define*

- *$u \sim_\alpha v$ iff* A *has a winning strategy in the game $\mathsf{G}_\alpha(u, v)$;*
- *$u \leq_\alpha v$ for $\alpha > 0$, iff* A *has a winning strategy for the game $\mathsf{G}'_\alpha(u, v)$;*
- *$u \leq_0 v$ iff $u(*) \geq v(*)$.*

We can in fact realise these relations as recursively generated from $\sim_0$. For any $u : P \to L$ and $p \in P$, we use $u_p : \uparrow p \to L$ to denote u precomposed with the open inclusion $\uparrow p \hookrightarrow P$. Then by definition we have the following recursive relations:

Lemma 3.2. *The following holds for the relations defined above:*

- $u \sim_{n+1} v$ *iff* $\forall p \in P.\exists q \in Q.u_p \sim_n v_q$ *and vice versa;*
- $u \leq_{n+1} v$ *iff* $\forall p \in P.\exists q \in Q.u_p \sim_n v_q$*;*
- $u \sim_\infty v$ *iff* $\forall p \in P.\exists q \in Q.u_p \sim_\infty v_q$ *and vice versa;*
- $u \leq_\infty v$ *iff* $\forall p \in P.\exists q \in Q.u_p \sim_\infty v_q$*;*
- $u \sim_\infty v$ *iff* $u \sim_n v$ *for a sufficiently large* n*;*

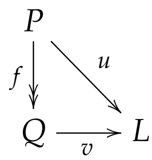
Proof. The first four conditions directly follow from the definition. For the last one, it holds because all our posets P, Q are *finite*. □

Lemma 3.3. *For all* $\alpha \in \mathbb{N} \cup \{\infty\}$*,* $\leq_\alpha$ *is a preorder and* $\sim_\alpha$ *is an equivalence relation, where* $u \sim_\alpha v$ *iff* $u \leq_\alpha v$ *and* $v \leq_\alpha u$*.*

Proof. The fact that $\leq_\alpha$ is a preorder follows from the fact that we can compose strategies. $u \sim_\alpha v$ iff $u \leq_\alpha v$ and $v \leq_\alpha u$ follows directly from Lemma 3.2. □

We proceed then to study more carefully these bisimulation relations in categorical terms. We first observe that $\sim_\infty$ and $\leq_\infty$ on preserved under precomposition of morphisms in $\mathbf{Por}_{\mathrm{f},*}$ in the following sense:

Lemma 3.4. $\leq_\infty$ *and* $\sim_\infty$ *satisfies descent for open maps and open surjections, respectively. Concretely, for any* $u : P \to L$ *and* $v : Q \to L$ *with an open surjection* $f : P \to Q$ *making the following diagramme commute,*

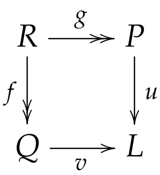


then $u \sim_\infty v$*; if* f *is only open, then* $u \leq_\infty v$*.*

Proof. It in fact suffices to show the case for open surjections, since any open map factors as an open surjection followed by an open inclusion, and in the definition of $\leq_\infty$ we restrict the game to play from P. But for open surjections, the commuting diagramme implies that it will be a p-morphism for the two Kripke models. □

Using this, we can give the following categorical characterisation of the two infinitary relations, which ultimately explains the relation of these bisimulation games and sheaves:

Proposition 3.5. *Given* $u : P \to L$ *and* $v : Q \to L$*,* $u \sim_\infty v$ *iff there exists covers* $f : R \twoheadrightarrow Q$ *and* $g : R \twoheadrightarrow P$ *in* $\mathbf{Por}_{\mathrm{f},*}$*, making the following diagramme commute,*



The weaker condition $u \leq_\infty v$ *holds iff* g *can be made open surjective while* f *only open.*

Proof. By Lemma 3.4, the if direction is evident. On the other hand, suppose $u \sim_\infty v$, we can take R to be the following rooted poset,

$$R := \{ (p, q) \mid u_p \sim_\infty v_q \}.$$

This is rooted because by definition, $u_* = u \sim_\infty v = v_*$, and the two projections are open surjections exactly follows from the third condition in Lemma 3.2. The characterisation for $u \leq_\infty v$ again immediately follows from this. □

Corollary 3.6. *For any subpresheaf X of $\underline{L}$ for some poset L, X is a sheaf in* $\mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J)$ *iff X is closed under $\sim_\infty$, i.e. if $u \in X(P)$ and $u \sim_\infty v$, then $v \in X(Q)$.*

Proof. If X is a sheaf with $u \in X(P)$ and $u \sim_\infty v$, by Proposition 3.5, there exists a diagramme

$$\begin{array}{ccc} R & \overset{g}{\twoheadrightarrow} & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\ Q & \underset{v}{\longrightarrow} & L \end{array}$$

In particular, $ug \in X(R)$, and by X being a sheaf, $v \in X(Q)$ as well since f is a cover.

On the other hand, suppose X is closed under $\sim_\infty$. To show it is a sheaf, it suffices to show it extends under open surjections:

$$\begin{array}{ccc} P & & \\ {\scriptstyle f}\downarrow & \searrow^{u} & \\ Q & \underset{v}{\twoheadrightarrow} & L \end{array}$$

Suppose $u \in X(P)$ and f is an open surjection in $\mathbf{Por}_{\mathrm{f},*}$. Since $\underline{L}$ is a sheaf, there exists a unique v extending it as shown above. By Lemma 3.4, $u \sim_\infty v$, thus $v \in X(Q)$ as well, which implies X is a sheaf. □

4. Sheaf Representation of Free Heyting Algebras

In the previous section, we have shown that for any poset L, the complete Heyting algebra $\mathrm{Sub}(\underline{L})$ in the sheaf topos $\mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J)$ are exactly those subpresheaves of $\underline{L}$ closed under the infinitary bisimulation relation $\sim_\infty$. However, if we want to represent Heyting algebras, we need to study *finitary* elements. In particular, our goal in this section is to show that the free Heyting algebra generated by a finite distributive lattice $\mathcal{D}(L)$ dual to the finite poset L, faithfully embeds into $\mathrm{Sub}(\underline{L})$,

$$\mathcal{H}(L) \rightarrowtail \mathrm{Sub}(\underline{L}),$$

making $\mathcal{H}(L)$ a sub Heyting algebra of $\mathrm{Sub}(\underline{L})$. In particular, elements in $\mathcal{H}(L)$ corresponds to certain finitary subsheaves in $\mathrm{Sub}(\underline{L})$.

Remark 4.1. *Caveat Lector!* Our notation $\mathcal{H}(L)$ could potentially be misleading: If L is a *discrete* poset, viz. a finite set, $\mathcal{H}(L)$ will *not* be the free Heyting algebra generated by elements in L in the algebraic sense, but freely generated by the *Boolean* algebra $\mathcal{D}(L) \cong \mathcal{P}(L)$, the power set of L.

If $\overline{x}$ is a set of variables whose length is equal to the cardinality of L, then our notation $\mathcal{H}(\overline{x})$ used in Section 1, which corresponds to the free Heyting algebra on $|\overline{x}|$-many generators, should be freely generated by the free distributive lattice $\mathcal{D}(\overline{x})$ on $|\overline{x}|$-many generators. However, $\mathcal{D}(\overline{x})$ is *not* isomorphic to $\mathcal{P}(L)$ unless L is empty. Hence from now on, if we want to denote the free generation in the set-theoretic sense, we will always use the notation $\mathcal{H}(\overline{x})$ or $\mathcal{D}(\overline{x})$ for a list of variables.

More precisely, our notation choice signifies the important shift of focus. The philosophy behind this is that, to present finitely presented Heyting algebras via the adjunction $\mathbf{HA}_{\mathrm{fp}} \rightleftarrows \mathbf{Set}$ is in some sense *too rigid*: We have too few free objects to play with. On the other hand, if instead we look at the adjunction $\mathbf{HA} \rightleftarrows \mathbf{DL}$, the framework becomes more flexible in the sense that there are more ways to represent a finitely presented Heyting algebra, because now we have more free objects. Notice that $\mathcal{H}(L)$ is *not* free w.r.t. any *set* of generators unless L is the powerset of some finite set. This more flexible framework makes it easier to study the categorical properties of the category of finitely presented Heyting algebras, which is well summarised in this sheaf-theoretic approach. ◊

Before that, let us recall the basic Heyting operations on $\mathrm{Sub}(\underline{L})$. Since J is a regular topology on $\mathbf{Por}_{\mathrm{f},*}$, both joins and meets in $\mathrm{Sub}(\underline{L})$ are point-wise: For any family of subsheaves $\{X_i\}_{i\in I}$ of $\underline{L}$ and any $P \in \mathbf{Por}_{\mathrm{f},*}$,

$$\left(\bigwedge_{i\in I} X_i\right)(P) = \bigcap_{i\in I} X_i(P), \quad \left(\bigvee_{i\in I} X_i\right)(P) = \bigcup_{i\in I} X_i(P).$$

The implication in $\mathrm{Sub}(\underline{L})$ is computed as follows,

$$(X \Rightarrow Y)(P) = \{\, u : P \to L \mid \forall f : Q \to P \in \mathbf{Por}_{\mathrm{f},*}.uf \in X(Q) \Rightarrow uf \in Y(Q) \,\}.$$

Our strategy is to characterise the image of $\mathcal{H}(L)$ using the bisimulation relation we have defined in the previous section. Inspired by Corollary 3.6, we give the following definition:

Definition 4.2. *For any $\alpha \in \mathbb{N} \cup \{\infty\}$, we say a subpresheaf X of $\underline{L}$ is an n-subsheaf if it is closed under $\sim_n$, i.e. if $u \in X(P)$ and $u \sim_n v$, then $v \in X(Q)$.*

In this definition, an ∞-subsheaf of $\underline{L}$ is the same as a subsheaf of $\underline{L}$; in particular, for smaller n, $\sim_n$ is larger, hence n-subsheaf will be a stronger notion. For any $\alpha \in \mathbb{N} \cup \{\infty\}$, we let $\mathrm{Sub}_\alpha(\underline{L})$ be the subposet of α-sheaves in $\mathrm{Sub}(\underline{L})$. In particular, $\mathrm{Sub}_\infty(\underline{L}) = \mathrm{Sub}(\underline{L})$. By the fact that meets and joins are computed point-wise, and by the definition of α-sheaves, $\mathrm{Sub}_\alpha(\underline{L})$ are in fact closed under arbitrary meets and joins in $\mathrm{Sub}(\underline{L})$.

Another important poset, which we denote as $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$, is the union of all finite subsheaves,

$$\mathrm{Sub}_{\mathbb{N}}(\underline{L}) := \bigcup_{i\in\mathbb{N}} \mathrm{Sub}_n(\underline{L}).$$

Our first goal is to show $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$ is a sub Heyting algebra of $\mathrm{Sub}(\underline{L})$.

Lemma 4.3. $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$ *is closed under finite meets and joins in* $\mathrm{Sub}(\underline{L})$.

Proof. Given a finite family $\{X_i\}$ of elements in $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$, we can find a large enough n such that all X_i are n-sheaves, hence so is their point-wise join and meets. $\square$

Remark 4.4. $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$ in general is not complete, because for an infinite family there could be no bound un the index of these sheaves. $\lozenge$

The more interesting result is that $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$ is closed under implication:

Lemma 4.5. *If $X, Y \in \mathrm{Sub}(\underline{L})$ are n-subsheaves, then $X \Rightarrow Y$ is an $n{+}1$-subsheaf*

Proof. Suppose $u : P \to L$ lies in $X \Rightarrow Y$, and let $u \sim_{n+1} v$. To show $v \in X \Rightarrow Y$, it suffices to show for any $f : R \to Q$ such that $vf \in X$, we would also have $vf \in Y$. We reason as follows:

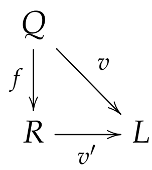
$$\begin{aligned} vf \leq_\infty v &\Rightarrow vf \leq_{n+1} u \\ &\Rightarrow \exists p \in P. vf \sim_n u_p \\ &\Rightarrow u_p \in X \\ &\Rightarrow u_p \in Y \\ &\Rightarrow vf \in Y \end{aligned}$$

The first because $v \sim_{n+1} u$; the second holds by Lemma 3.2; the third by the fact that X is an n-subsheaf, and the forth by the fact that $u \in X \Rightarrow Y$; the final step again holds since Y is an n-sheaf. $\square$

Corollary 4.6. $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$ *is a sub Heyting algebra of* $\mathrm{Sub}(\underline{L})$*.*

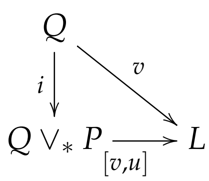
To study these n-sheaves, we first need analogous results like Proposition 3.5 to characterise these bisimulation relation categorically. Although a full characterisation is not possible, we have the following useful results:

Lemma 4.7. *For any $u : P \to L$ and $v : Q \to L$, $v \leq_n u$ iff we can find an open map*



such that $v' \sim_n u$.

Proof. The if direction is easy. By Lemma 3.4, we know $v \leq_\infty v'$, hence if $v' \sim_n u$ then $v \leq_n u$. On the other hand, suppose $v \leq_n u$, we construct R as follows. Let R be the rooted wedge $Q \vee_* P$ of Q and P. Concretely, R is the poset obtained by putting Q above the root of P, but inserting no further relations. There is an evident diagramme as follows,



Notice that $[v, u]$ is monotone relies on the fact that $v(*) \geq u(*)$. Furthermore, we do have $u \sim_n [v, u]$: For any rounds, either E chooses an element in P, which we can respond for A with the same element in $Q \vee_* P$; the same goes if E chooses an element in the copy of P in $Q \vee_* P$. The only case we need to verify is when E first choose an element in the copy of Q in $Q \vee_* P$, but that follows from the fact that $v \leq_n u$. □

In particular, Lemma 4.7 suggests that the relation $\leq_n$ is exactly the relation $\sim_n$ closed under precomposition with morphisms in $\mathbf{Por}_{\mathrm{f},*}$. This has direct consequences of our characterisation of n-sheaves, in that the following *a priori* stronger definition of n-sheaves results in the same notion:

Lemma 4.8. *$X \in \mathrm{Sub}(\underline{L})$ is an n-subsheaf iff it is downward closed under $\leq_n$, i.e. if $u \in X(P)$ and $v \leq_n u$, then $v \in X(Q)$.*

Proof. The if direction is trivial. For the only if direction, suppose X is an n-sheaf, with $u \in X(P)$ and $v \leq_n u$. By Lemma 4.7, we can find v' and an open map f with $v = v'f$ such that $v' \sim_n u$. X being an n-sheaf implies $v' \in X$, hence $v \in X$ as well by functoriality. □

Using Lemma 4.8, we can in fact find special elements in the Heyting algebra $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$. For any $u : P \to L$ and any $\alpha \in \mathbb{N} \cup \{\infty\}$, let $\downarrow_\alpha u$ be the following subpresheaf:

$$\downarrow_\alpha u(Q) := \{\, v : Q \to L \mid v \leq_\alpha u \,\}.$$

Notice that $\downarrow_\alpha u$ is indeed a presheaf because if $v \leq_\alpha u$, then for any open f, $vf \leq_\infty v$ implies $vf \leq_\alpha u$, hence $vf \in \downarrow_\alpha u$ as well. Lemma 4.7 then implies that for any $n \in \mathbb{N}$, $\downarrow_n u$ is in fact an n-sheaf. In fact, $\downarrow_\infty u$ is also an n-sheaf for some n, because the domain P is finite.

The importance of these elements lies in the fact that they generate each stratefied subposet $\mathrm{Sub}_\alpha(\underline{L})$ via joins: For any α-subsheaf X, evidently we have

$$X = \bigvee_{u \in X} \downarrow_\alpha u.$$

As a first step to show $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$ is the free Heyting algebra generated by L, we need to embed the free distributive lattice generated by L into $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$. Let $\mathcal{D}(L)$ be the lattice of upward closed subsets of L. When L is finite, this is indeed the free distributive lattice on L. We show $\mathcal{D}(L)$ is exactly the poset of 0-sheaves:

Lemma 4.9. *For any poset L, there is an isomorphism*

$$\iota : \mathcal{D}(L) \cong \mathrm{Sub}_0(\underline{L}),$$

where for each upward closed subset U of L, we have

$$\iota_U(P) := \{\, u : P \to L \mid u(*) \in U \,\}.$$

Proof. First notice that ι_U is indeed a 0-sheaf for any U: Suppose $u \in \iota_U$ and $v \leq_0 u$, viz. $v(*) \geq u(*)$, then $v(*) \in U$ as well. On the other hand, given a 0-sheaf X, we can construct an upward closed subset U_X of L as follows:

$$U_X := \{\, x \in L \mid \exists u \in X.u(*) \leq x \,\}.$$

We show the two constructions are inverses to each other. On one hand,

$$\begin{aligned} v \in \iota_{U_X} &\Leftrightarrow v(*) \in U_X \\ &\Leftrightarrow \exists u \in X.u(*) \leq v(*) \\ &\Leftrightarrow \exists u \in X.v \leq_0 u \\ &\Leftrightarrow v \in X. \end{aligned}$$

The final step uses the fact that X is a 0-sheaf. On the other hand,

$$\begin{aligned} x \in U_{\iota_U} &\Leftrightarrow \exists u \in \iota_U.u(*) \leq x \\ &\Leftrightarrow \exists u : P \to L.u(*) \in U \wedge u(*) \leq x \\ &\Leftrightarrow x \in U \end{aligned}$$

Hence, the above shows that $\iota_{(-)}$ and $U_{(-)}$ are inverses to each other, and thus $\mathcal{D}(L)$ is isomorphic to $\mathrm{Sub}_0(\underline{L})$. □

The above then shows that if we want to show $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$ is the free Heyting algebra generated by $\mathcal{D}(L)$, then we must first show every element in $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$ can ultimately be written as an implication of elements with smaller index. This crucially relies on L being finite, because the following simple but crucial fact is used:

Lemma 4.10. *For finite poset L and $n \in \mathbb{N}$, $\sim_n$ has only finitely many equivalence classes.*

Proof. We prove by induction. For 0 this follows from L being finite. For $n+1$, notice that $u \not\sim_{n+1} v$ iff $\exists p \in P.\forall q \in Q.u_p \not\sim_n v_q$ by Lemma 3.2. However, for a fixed u, there are only finitely many u_p since P is finite; and the fact that there are only finitely many $\sim_n$ equivalence classes implies that there can only be finitely many ways to distinguish u using $\sim_{n+1}$, hence again $\sim_{n+1}$ is finite. □

Using this fact, for any fixed $n \in \mathbb{N}$ the poset $\mathrm{Sub}_n(\underline{L})$ will be finite. Combined with our canonical representation of any n-subsheaf as a join of elements of the form $\downarrow_n u$, we can first prove:

Lemma 4.11. *The uniquely induced map of Heyting algebras $\mathcal{H}(L) \to \mathrm{Sub}_{\mathbb{N}}(\underline{L})$ below is an surjective,*

$$\begin{array}{ccc} \mathcal{D}(L) & & \\ \downarrow & \overset{\iota}{\searrow} & \\ \mathcal{H}(L) & \twoheadrightarrow & \mathrm{Sub}_{\mathbb{N}}(\underline{L}) \end{array}$$

Proof. For surjectivity, it suffices to show every element $\downarrow_n u$ lies in the image. This can be proven inductively. For $n = 0$, this is taken care of by Lemma 4.9. For $n+1$, it suffices to show

$$\downarrow_{n+1} u = \bigwedge_{\forall p.v \not\sim_n u_p} \left(\downarrow_n v \Rightarrow \bigvee_{v \not\leq_n w} \downarrow_n w \right).$$

On one hand,

$$\begin{aligned}\downarrow_{n+1} u \leq & \bigwedge_{\forall p. v \not\sim_n u_p} \left(\downarrow_n v \Rightarrow \bigvee_{v \not\leq_n w} \downarrow_n w \right) \\ \Leftrightarrow & \forall p. v \not\sim_n u_p \Rightarrow \downarrow_{n+1} u \wedge \downarrow_n v \leq \bigvee_{v \not\leq_n w} \downarrow_n w.\end{aligned}$$

Given such v, for any $w \leq_{n+1} u$ and $w \leq_n v$, we show $v \not\leq_n w$. Otherwise, if $v \leq_n w$ then there exists some q that $v \sim_n w_q$. Since $w \leq_{n+1} u$, this implies there exists some p that $w_q \sim_n u_p$, thus $v \sim_n u_p$, contradictory. Hence, the above relation holds.

On the other hand, suppsoe for some x we have it belongs to the RHS. We must show $x \leq_{n+1} u$. If not, then there exists some q such that $x_q \not\sim_n u_p$ for all p. Notice that $x \notin \downarrow_n x_q \Rightarrow \bigvee_{x_q \not\leq_n w} \downarrow_n w$, otherwise $x_q \in \downarrow_n x_q$ would imply $x_q \in \bigvee_{x_q \not\leq_n w} \downarrow_n w$, but the latter cannot be true. Thus, $x \leq_{n+1} u$, hence $x \in \downarrow_{n+1} u$. □

To prove the induced map is *injective*, we need more results on Heyting algebras. The most crucial property is the following, which is a direct consequence of finite model property of intuitionistic logic:

Lemma 4.12. *Every finitely presented Heyting algebra can be embeded into a product of finite Heyting algebras.*

Proof. Suppose we have a finitely presented Heyting algebra $\mathcal{H}(\overline{x})/E(\overline{x})$, and we think of it as the intuitionistic calculus of variables $\overline{x}$ quotiented out by a formula φ_E. Suppose two formulas ψ_0, ψ_1 are not equivalent over φ_E, i.e.

$$\nvdash_{\mathbb{H}} \varphi_E \to (\psi_0 \leftrightarrow \psi_1).$$

By completeness of intuitionistic logic over finite Kripke models, we can find a Kripke model which validates φ_E but differences ψ_0 and ψ_1. By Duality, this implies that there exists a finite Heyting algebra A and an algebraic map

$$\mathcal{H}(\overline{x})/E(\overline{x}) \to A$$

where ψ_0, ψ_1 are distinguished in A. Hence, there is an embedding

$$\mathcal{H}(\overline{x})/E(\overline{x}) \rightarrowtail \prod_{A \text{ finite quotients}} A,$$

where A ranges over all finite images of $\mathcal{H}(\overline{x})/E(\overline{x})$. □

Remark 4.13. From an algebra point of view, the above result is actually equivalent to the fact that there is an equivalence of categories

$$\mathrm{Pro}(\mathbf{HA}_\mathrm{f}) \simeq \mathbf{Stone_{HA}},$$

i.e. the pro-completion of the category of finite Heyting algebras is equivalent to Stone Heyting algebras. This holds more generally for monoids, groups, rings, lattices, distributive lattices, and so on. See Johnstone's *Stone Space* for more information. ◊

Hence, to show injectivity of $\mathcal{H}(L) \to \mathrm{Sub}_{\mathbb{N}}(\underline{L})$, we only need to show that this map factors through the embedding of $\mathcal{H}(L)$ into a product of finite Heyting algebras, because $\mathcal{H}(L)$ is finitely presented,

$$\begin{array}{ccc} & \nearrow & \mathrm{Sub}_{\mathbb{N}}(\underline{L}) \\ & & \downarrow \\ \mathcal{H}(L) & \longrightarrow & \prod A \end{array}$$

Lemma 4.14. *The uniquely induced map $\mathcal{H}(L) \to \mathrm{Sub}_{\mathbb{N}}(\underline{L})$ is also injective.*

Proof. To better compare with our chosen site $(\mathbf{Por}_{\mathrm{f},*}, J)$, using the duality of finite Heyting algebras and Lemma 2.4 we can embed $\mathcal{H}(L)$ as follows,

$$\mathcal{H}(L) \rightarrowtail \prod_i \mathcal{D}(P_i)$$

where each P_i is a finite rooted poset. Then our goal is to construct for each Heyting algebra morphism $a_i : \mathcal{H}(L) \to \mathcal{D}(P_i)$ a corresponding $b_i : \mathrm{Sub}_{\mathbb{N}}(\underline{L}) \to \mathcal{D}(P_i)$. Since $\mathcal{H}(L)$ is freely generated over $\mathcal{D}(L)$, it suffices to make the following diagramme commute,

$$\begin{array}{ccc} \mathcal{D}(L) & \xrightarrow{\iota} & \mathrm{Sub}_{\mathbb{N}}(\underline{L}) \\ \downarrow & & \downarrow b_i \\ \mathcal{H}(L) & \xrightarrow[a_i]{} & \mathcal{D}(P_i). \end{array}$$

By duality, the map $\mathcal{D}(L) \rightarrowtail \mathcal{H}(L) \to \mathcal{D}(P_i)$ corresponds to a unique monotone map $u_i : P_i \to L$. We may then construct the map b_i as follows:

$$b_i(X) := \{\, p \in P_i \mid u_{i,p} \in X \,\}.$$

This subset is upward closed: If $p \in b_i(X)$ and $p \leq q$ in P_i, then $u_{i,q} \leq_\infty u_{i,p}$, thus X being a sheaf implies $u_{i,q} \in X$, hence $q \in b_i(X)$ as well. Now for any $U \in \mathcal{D}(L)$, we calculate

$$\begin{aligned} b_i \iota_U &= \{\, p \in P_i \mid u_{i,p} \in \iota_U \,\} \\ &= \{\, p \in P_i \mid u_i(*) \in U \,\} \\ &= \{\, p \in P_i \mid u(p) \in U \,\} \\ &= u_i^{-1}(U) \end{aligned}$$

which exactly implies the above diagramme commutes, because the algebraic map $\mathcal{D}(L) \to \mathcal{D}(P_i)$ corresponding to $u_i : P_i \to L$ is exactly the inverse image. ☐

Theorem 4.15. *For a finite distributive lattice $\mathcal{D}(L)$ dual to a finite poset L, the free Heyting algebra $\mathcal{H}(L)$ generated by $\mathcal{D}(L)$, and $\mathcal{D}(L)$ itself, has the following sheaf representation:*

$$\begin{array}{ccc} \mathcal{D}(L) & \xrightarrow{\cong} & \mathrm{Sub}_0(\underline{L}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{H}(L) & \xrightarrow[\cong]{} & \mathrm{Sub}_{\mathbb{N}}(\underline{L}) \end{array}$$

5. A LOGICAL DESCRIPTION OF THE SHEAF REPRESENTATION

Notice that, up to this point, our proof for the representation result Theorem 4.15 is *purely algebraic and game-theoretic*. In particular, we have never even used the fact that the semantics of intuitionistic formulas are invariant under the bisimulation relation we have used extensively, although this is evidently our motivation in mind. This in some sense suggests that the previous development is *independent* from the logical investigation, but also obscures some logical intuition we have in mind. In this section, we make this logical perspective clearer.

If we think of $\mathcal{D}(L)$ as a propositional coherent theory, whose elements will now be denoted by $\alpha, \beta, \gamma \cdots$, then by duality L is the poset of *models* of $\mathcal{D}(L)$. In particular, now we write for any $x \in L$ and $\alpha \in \mathcal{D}(L)$,

$$x \models \alpha \Leftrightarrow x \in \alpha.$$

The order in L now can be described in this logical form as

$$x \leq y \in L \Leftrightarrow \forall \alpha \in \mathcal{D}(L), x \models \alpha \Rightarrow y \models \alpha.$$

A monotone map $u : P \to L$ then could then be thought of as assigning to each $p \in P$ an evaluation of each coherent formula in $\mathcal{D}(L)$, viz. a model in L, in a compatible way.

The difference of the above description of the Kripke model on P and the usual one is that, the assignment of evaluations $u : P \to L$ already takes care of all the evaluation of coherent formulas coming from $\mathcal{D}(L)$. This again reflects our perspective on generating free Heyting algebras by distributive lattices, rather than by a set of generator.

Now our first representation given by Lemma 4.9

$$\mathcal{D}(L) \cong \mathrm{Sub}_0(\underline{L})$$

actually shows that a coherent formula α in the distributive lattice $\mathcal{D}(L)$ can be thought of as the family of Kripke models $u : P \to L$ whose root validates α, viz. $u(*) \models \alpha$, or $u(*) \in \alpha$. The lattice operation in $\mathcal{D}(L)$ then can be taken as the intersections and unions of family of models. This is simply a very redundant way of representing formulas in $\mathcal{D}(L)$, since they are essentially upward closed subsets of L, and we are just collecting all those Kripke models whose root is assigned an element in this upward closed set.

However, the crucial thing about the above seemingly redundant representation is that we can represent the free Heyting algebra $\mathcal{H}(L)$ generated by $\mathcal{D}(L)$ in the same way. Our first task is to describe the semantic relation for these slightly more general kinds of Kripke models:

Definition 5.1. *For any $\varphi \in \mathcal{H}(L)$ and any Kripke model $u : P \to L$, we describe the semantics relation $u \models \varphi$, more precisely satisfaction at the root of P, in the following inductive way:*

- *If $\varphi = \alpha \in \mathcal{D}(L)$, then*

$$u \models \varphi \Leftrightarrow u(*) \models \varphi.$$

- *If $\varphi = \psi \to \chi$, then*

$$u \models \varphi \Leftrightarrow \forall p \in P. u_p \models \psi \Rightarrow u_p \models \chi.$$

Using this semantic relation, we can in fact provide a more concrete description of what the isomorphism $\mathcal{H}(L) \cong \mathrm{Sub}_{\mathbb{N}}(\underline{L})$ is. Again, the idea is that a formula φ should corresponds to the family of models which validates it at the root. But to show this definition works, we first need to realise that our bisimulation relation indeed plays well with the semantic relation we give in Definition 5.1.

We let $\mathcal{H}_n(L)$ be the set of elements $\varphi \in \mathcal{H}(L)$ which can be written as an intuitionistic formula over $\mathcal{D}(L)$ with depth at most n. Then we have the following bisimulation invariance result:

Lemma 5.2. *For any $\varphi \in \mathcal{H}_n(L)$, if $u \sim_n v$ then $u \models \varphi$ iff $v \models \varphi$.*

Proof. We prove this again by induction. The base case is trivial. Suppose $\varphi = \psi \to \chi$ with $\psi, \chi \in \mathcal{H}_n(L)$, $u \leq_{n+1} v$ and $u \models \psi$, then we have

$$v \models \varphi \Leftrightarrow \forall q. v_q \models \psi \Rightarrow v_q \models \chi.$$

However, since $u \sim_{n+1} v$, for any q we can find p such that $u_p \sim_n v_q$, thus

$$v_q \models \psi \Rightarrow u_p \models \psi \Rightarrow u_p \models \chi \Rightarrow v_q \models \chi.$$

The first and last implication follows from inductive hypothesis, and the middle one holds by our assumption $u \models \varphi$. The other direction follows completely symmetrically. □

And the asymmetric version:

Lemma 5.3. *For any $\varphi \in \mathcal{H}_n(L)$, if $v \leq_n u$, then $u \models \varphi$ implies $v \models \varphi$.*

Proof. Do induction again by using Lemma 5.2. Left as an Exercise. □

Equipped with these invariance result, we can finally proceed as we would like. For any $\varphi \in \mathcal{H}(L)$, let X_φ be the following subsheaf

$$X_\varphi = \{ u : P \to L \mid u \models \varphi \}.$$

First notice that X_φ is a subpresheaf of $\underline{L}$, because if we have $u \in X_\varphi$ and f open, then $uf \leq_\infty u$ hence by Lemma 5.3, $uf \models \varphi$. In fact, Lemma 5.3 further shows that if $\varphi \in \mathcal{H}_n(L)$, then X_φ is an n-subsheaf. This means that the assignment $\varphi \mapsto X_\varphi$ gives us a map

$$X_- : \mathcal{H}(L) \to \mathrm{Sub}_{\mathbb{N}}(\underline{L}),$$

or in fact a map

$$X_- : \mathcal{H}_n(L) \to \mathrm{Sub}_n(\underline{L})$$

for any $n \in \mathbb{N}$. This assignment evidently preserves meets and joins, since on the sheaf level they are computed as point-wise intersections and unions. Hence the crucial thing to observe that is it preserves implication as well:

Lemma 5.4. *The assignment $X_- : \mathcal{H}(L) \to \mathrm{Sub}_{\mathbb{N}}(\overline{L})$ is a Heyting algebra morphism.*

Proof. As mentioned above, we only need to show it preserves implication, i.e. we need to show $X_{\psi\to\chi} = X_\psi \Rightarrow X_\chi$. Recall the implication in $\mathrm{Sub}_{\mathbb{N}}(\underline{L})$ is computed as follows,

$$X_\psi \Rightarrow X_\chi = \{\, u : P \to L \mid \forall f. uf \in X_\psi \Rightarrow uf \in X_\chi \,\}.$$

On one hand, if $u \in X_{\psi\to\chi}$, i.e. $u \models \psi \to \chi$, then for any f in $\mathbf{Por}_{\mathrm{f},*}$, $uf \leq_\infty u$ implies that $uf \models \psi \to \chi$. Hence, if $uf \models \psi$, then $uf \models \chi$ as well, which means $u \in X_\psi \Rightarrow X_\chi$.

On the other hand, suppose $u \in X_\psi \Rightarrow X_\chi$, For any p, if $u_p \models \psi$, then by definition $u_p \in X_\psi$, hence $u_p \in X_\chi$ as well, whcih means $u_p \models \psi \to \chi$. This shows that $X_{\psi\to\chi} = X_\chi \Rightarrow X_\psi$. □

Theorem 5.5. *The isomorphism constructed in Theorem 4.15 is given by*

$$X_- : \mathcal{H}(L) \to \mathrm{Sub}_{\mathbb{N}}(\overline{L}),$$

i.e. assigning each intuitionistic formula to the set of models which validates it at the root. In fact, the logical formulation gives as the stratefied isomorphisms for any $n \in \mathbb{N}$,

$$X_- : \mathcal{H}_n(L) \to \mathrm{Sub}_n(\overline{L}).$$

Proof. We only need to show that our X_- coincide with the map given in Theorem 4.15. By the universal property of $\mathcal{H}(L)$, we only need to show they coincide for $\mathcal{D}(L)$, which we have already verified at the beginning of this section. □

As an application in logic, the above result as a consequence of the algebraic and game-theoretic representation given in Theorem 4.15, now implies the following much stronger bisimulation invariance result, i.e. two models are n-bisimular *iff* they agrees with formulas upto depth n:

Theorem 5.6. *For any $u, v \in \underline{L}$, $u \sim_n v$ iff for all $\varphi \in \mathcal{H}_n(L)$, $u \models \varphi$ iff $v \models \varphi$.*

Proof. Only the if direction is non-trivial. Suppose $u \not\sim_n v$, then either $v \notin \downarrow_n u$ or $u \notin \downarrow_n v$. Suppose the former holds. Under the isomorphism $\mathcal{H}_n(L) \cong \mathrm{Sub}_n(\underline{L})$, the n-subsheaf $\downarrow_n u$ corresponds to a formula $\varphi \in \mathcal{H}_n(L)$. Since $v \notin \downarrow_n u$, it follows that $v \not\models \varphi$ by our logical representation Theorem 5.5. Hence we have found a formula witnessing the difference of u, v. The other case is completely symmetric. □

6. Sheaf Representation of the Syntactic Heyting Site

One important fact about finitely presented Heyting algebras $\mathcal{H}(\overline{x})/E(\overline{x})$ is that, algebraically, we can find a term $\varphi \in \mathcal{H}(\overline{x})$ such that

$$\mathcal{H}(\overline{x})/E(\overline{x}) \cong \downarrow \varphi.$$

In particular, if $E(\overline{x})$ is given by $\{\varphi_i = \varphi_i'\}$, then we can take φ to be

$$\varphi := \bigwedge_i (\varphi_i \leftrightarrow \varphi_i'),$$

and the quotient is realised as taking conjunction with φ,

$$\varphi^* : \mathcal{H}(\overline{x}) \to \downarrow \varphi \cong \mathcal{H}(\overline{x})/E(\overline{x}).$$

This combined with our representation given by Theorem 4.15, implies we can view a finitely presented Heyting algebras as certain finite subsheaf X of $\underline{L}$, for some poset L, where we think of X as representing the Heyting algebra $\mathcal{H}(L)/\varphi_X$, under the isomorphism

$$\varphi_- : \mathrm{Sub}_{\mathbb{N}}(\underline{L}) \cong \mathcal{H}(L).$$

This way, we can think of objects in $\mathcal{H}$ as objects in $\mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J)$.

However, the easier way is not to embed $\mathcal{H}$ into $\mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J)$ directly, but to embed $\mathbf{HA}_{\mathrm{fp}}^{\mathrm{op}}$ instead. The reason is that our preferred presentation of a finitely presented Heyting algebra now is *not* of the form $\mathcal{H}(\overline{x})/E(\overline{x})$ anymore, but as quotients of $\mathcal{H}(L)$ for some finite poset L. However, the description of $\mathbf{HA}_{\mathrm{fp}}^{\mathrm{op}}$ allows us to focus on the semantic information, rather than syntactic representations.

In particular, for any finitely presented Heyting algebra H, we consider the following presheaf Φ_H on $\mathbf{Por}_{\mathrm{f},*}$,

$$\Phi_H(P) := \mathbf{HA}_{\mathrm{fp}}(H, \mathcal{D}(P)).$$

In other words, Φ_H can be viewed as the collection of models of H. From this description and our logical representation indicated by Theorem 5.5, the following result should be expected:

Lemma 6.1. *If H is isomorphic to the quotient $\downarrow \varphi_X$ of $\mathcal{H}(L)$ for some finite poset L and some finitary subsheaf X of L, then $\Phi_H \cong X$.*

Proof. For any finite rooted poset P, we have a sequence of natural isomorphisms

$$\begin{aligned}
\Phi_H(P) &\cong \mathbf{HA}_{\mathrm{fp}}(H, \mathcal{D}(P)) \\
&\cong \mathbf{HA}_{\mathrm{fp}}(\downarrow \varphi_X, \mathcal{D}(P)) \\
&\cong \{ f \in \mathbf{HA}_{\mathrm{fp}}(\mathcal{H}(L), \mathcal{D}(P)) \mid f\varphi_X = 1 \} \\
&\cong \{ f \in \mathbf{DL}(\mathcal{D}(L), \mathcal{D}(P)) \mid f\varphi_X = 1 \} \\
&\cong \{ u : P \to L \mid u \models \varphi_X \} \\
&\cong X(P).
\end{aligned}$$

The only non-trivial isomorphisms are the last two. The second to last one holds by duality of finite distributive lattices, and the last one holds by our representation indicated by Theorem 5.5. □

Corollary 6.2. *We have a faithful functor*

$$\Phi : \mathbf{HA}_{\mathrm{fp}}^{\mathrm{op}} \rightarrowtail \mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J).$$

Proof. Lemma 6.1 implies that Φ_H is a finitary subsheaf of some $\underline{L}$, which in particular is a sheaf, hence Φ is a well-defined functor. Its faithfulness is implied by Lemma 4.12. □

Thus, we have identified $\mathbf{HA}_{\mathrm{fp}}^{\mathrm{op}}$, or $\mathcal{H}$, as a subcategory of $\mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J)$. The remaining task is to provide an internal account of those morphisms coming from $\mathbf{HA}_{\mathrm{fp}}^{\mathrm{op}}$, so that we can more easily show they are closed under the Heyting category structure existing in $\mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J)$.

By definition, morphisms coming from $\mathbf{HA}_{\mathrm{fp}}^{\mathrm{op}}$ or $\mathcal{H}$ must be *definable* by terms of Heyting algebras. Notice that we only need to study definable morphisms between $\underline{L}$ and $\underline{M}$, or $\Phi_{\mathcal{H}(L)}$ and $\Phi_{\mathcal{H}(M)}$, for two finite posets L, M, since the definable morphisms between their quotients are exactly definable morphisms between them which respects the quotients.

To characterise these definable morphisms, we start by studying the behaviors of $\Phi_t : \Phi_{\mathcal{H}(L)} \to \Phi_{\mathcal{H}(M)}$ for some algebraic map $t : \mathcal{H}(M) \to \mathcal{H}(L)$. Lemma 6.1 suggests that there is a bijective correspondence between $\underline{L}(P)$ and Heyting algebra morphisms $\mathcal{H}(L) \to \mathcal{D}(P)$. For any $u : P \to L$, we will write $\overline{u} : \mathcal{H}(L) \to \mathcal{D}(P)$ as the corresponding algebraic map, which is uniquely induced by

$$\begin{array}{ccc} \mathcal{D}(L) & & \\ \downarrow & \searrow^{u^{-1}} & \\ \mathcal{H}(L) & \underset{\overline{u}}{\dashrightarrow} & \mathcal{D}(P) \end{array}$$

Then Φ_t acts as precomposition with $t : \mathcal{H}(M) \to \mathcal{H}(L)$. This description implies the following crucial observation:

Lemma 6.3. *For any $\varphi \in \mathcal{H}(M)$, we have*

$$\Phi_t^* X_\varphi \cong X_{t\varphi}.$$

In other words, the following diagramme is a pullback,

$$\begin{array}{ccc} X_{t\varphi} & \longrightarrow & X_\varphi \\ \downarrow & & \downarrow \\ \underline{L} & \underset{\Phi_t}{\longrightarrow} & \underline{M} \end{array}$$

Proof. By definition,

$$\begin{aligned} u \in X_{t\varphi} &\Leftrightarrow u \models t\varphi \Leftrightarrow \overline{u} \models t\varphi \\ &\Leftrightarrow \overline{u} \circ t \models \varphi \Leftrightarrow \Phi_t(u) \models \varphi \\ &\Leftrightarrow \Phi_t(u) \in X_\varphi. \end{aligned}$$

This shows that $X_{t\varphi}$ is indeed the pullback of X_φ along Φ_t. □

This way, we have obtained a very simple description of definable morphisms:

Proposition 6.4. *A morphism $\eta : \underline{L} \to \underline{M}$ is definable, i.e. lies in the image of Φ, iff the pullback functor η^* restricts to one on finitary subsheaves,*

$$\eta^* : \mathrm{Sub}_{\mathbb{N}}(\underline{M}) \to \mathrm{Sub}_{\mathbb{N}}(\underline{L}).$$

Proof. This is essentially an easy consequence of Lemma 6.3, the faithfulness of Φ given in Corollary 6.2, and our representation Theorem 5.5. □

Again, this can be seen as a purely algebraic characterisation of the image of Φ. However, we are also interested in a logical characterisation. In order to do this, we first try to reformulate the semantic relation given in Definition 5.1 in algebraic terms:

Lemma 6.5. *For any $u : P \to L$ and $\varphi \in \mathcal{H}(L)$, $u \models \varphi$ iff $\overline{u}(\varphi) = P$.*

Proof. We prove this by induction. For the base caes if $\alpha \in \mathcal{D}(L)$, then by definition

$$u \models \alpha \Leftrightarrow u(*) \models \alpha \Leftrightarrow u(*) \in \alpha \Leftrightarrow u^{-1}(\alpha) = P.$$

For the inductive case, if $\varphi = \psi \to \chi$, then we have

$$\begin{aligned} u \models \psi \to \chi &\Leftrightarrow \forall p \in P.u_p \models \psi \Rightarrow u_p \models \chi \\ &\Leftrightarrow \forall p \in P.\overline{u_p}(\psi) = \uparrow p \Rightarrow \overline{u_p}(\chi) = \uparrow p \\ &\Leftrightarrow \forall p \in P.i_p^{-1}\overline{u}(\psi) = \uparrow p \Rightarrow i_p^{-1}\overline{u}(\chi) = \uparrow p \\ &\Leftrightarrow \forall p \in P.p \in \overline{u}(\psi) \Rightarrow p \in \overline{u}(\chi). \end{aligned}$$

where $i_p : \uparrow p \hookrightarrow P$ is the open inclusion of the principal upward closed subset on p. On the other hand, we know that

$$\overline{u}(\psi \to \chi) = \overline{u}(\psi) \Rightarrow \overline{u}(\chi) = \{\, p \in P \mid p \in \overline{u}(\psi) \Rightarrow p \in \overline{u}(\chi) \,\}.$$

This way, it follows that $u \models \psi \to \chi$ iff $\overline{u}(\psi \to \chi) = P$. □

In particular, using the bisimulation invariance result stated in Theorem 5.6, we can now define for algebraic maps $\overline{u} : \mathcal{H}(L) \to \mathcal{D}(P)$ and $\overline{v} : \mathcal{H}(L) \to \mathcal{D}(Q)$,

$$\overline{u} \sim_n \overline{v} \Leftrightarrow \forall \varphi \in \mathcal{H}_n(L), \overline{u} \models \varphi \Leftrightarrow \overline{v} \models \varphi \Leftrightarrow u \sim_n v.$$

This allows us to characterise definable morphisms as follows:

Proposition 6.6. *A natural transformation $\eta : \underline{L} \to \underline{M}$ is definable iff there exists some n, such that for any $u \sim_n v$ in $\underline{L}$, $\eta u \sim_0 \eta v$.*

Proof. For the only if direction, any such algebraic map t is uniquely determined by a map of distributive lattices

$$t : \mathcal{D}(M) \to \mathcal{H}(L),$$

where t can be viewed as a finite family of terms $\{t_\alpha\}_{\alpha \in \mathcal{D}(M)}$. In particular, there exists some $n \in \mathbb{N}$ that $t_\alpha \in \mathcal{H}_n(L)$ for all $\alpha \in \mathcal{D}(M)$. Now suppose $u \sim_n v$. If η is induced by t, then ηu would corresponds to the following algebraic map

$$\mathcal{D}(M) \xrightarrow{t} \mathcal{H}(L) \xrightarrow{\overline{u}} \mathcal{D}(P)$$

and similarly for ηv. Notice that we have

$$\eta u \sim_0 \eta v \Leftrightarrow \overline{u} \sim_0 \overline{u} \Leftrightarrow \forall \alpha \in \mathcal{D}(M).\overline{u}(t_\alpha) = P \Leftrightarrow \overline{v}(t_\alpha) = P.$$

Since $t_\alpha \in \mathcal{H}_n(L)$ and $\overline{u} \sim_n \overline{v}$, this indeed holds, hence $\eta u \sim_0 \eta v$.

For the if direction, by Proposition 6.4, we need to show that η^* takes finitary subsheaves to finitary subsheaves. Since η^* is a Heyting algebra morphism, it suffices to show that 0-subsheaves of $\underline{M}$ are taken to finitary subsheaves on $\underline{L}$. Given any $X \in \mathrm{Sub}_0(\underline{M})$, we show that $\eta^* X$ is an n-subsheaf. Given $u \in \eta^* X$ and $u \sim_n v$, by assumption on η we have $\eta u \sim_0 \eta v$. Since X is a 0-subsheaf, $\eta v \in X$, hence $v \in \eta^* X$. This implies $\eta^* X$ is an n-subsheaf. □

For a definable morphism $\eta : \underline{L} \to \underline{M}$, if it satisfies the condition stated in our logical characterisation given by Proposition 6.6 for some n, we will then say it is *n-definable*. The above proof actually shows that for an n-definable η,

$$\eta^* : \mathrm{Sub}_0(\underline{M}) \to \mathrm{Sub}_n(\underline{L}).$$

Combined with our logical representation in Theorem 5.5, for any $m \in \mathbb{N}$ we would then have

$$\eta^* : \mathrm{Sub}_m(\underline{M}) \to \mathrm{Sub}_{n+m}(\underline{L}).$$

This way, we have obtained an intrinsic characterisation of those morphisms lying inside the image of the representation functor $\Phi : \mathbf{HA}_{\mathrm{fp}}^{\mathrm{op}} \to \mathrm{Sh}(\mathbf{Por}_{\mathrm{f},*}, J)$.