Synthetic Quasi-Coherence for Coherent Theories

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Classifying topos

Let $\mathbb{T}$ be a geometric (coherent) theory ($\exists, \wedge, \bigvee / \vee$). It has a *classifying topos* $\mathbf{Set}[\mathbb{T}]$:

- $\mathbf{Set}[\mathbb{T}]$ classifies $\mathbb{T}$-models in topoi: $\mathbf{Mod}_{\mathbb{T}}(\mathcal{E}) \simeq \mathbf{Topoi}(\mathcal{E}, \mathbf{Set}[\mathbb{T}])$.
- The equivalence is generated by a *generic* $\mathbb{T}$*-model* $U_{\mathbb{T}}$ in $\mathbf{Set}[\mathbb{T}]$.
- For any *geometric sequent* σ in $\mathbb{T}$, $U_{\mathbb{T}} \models \sigma$ iff $\mathbb{T} \vdash \sigma$.

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Central question

What about non-geometric ($\forall, \to$ and higher-order) properties of $U_{\mathbb{T}}$?

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Central question (upgraded)

What are the possibilities and limitations of internal languages of topoi?

Some curious examples

The theory of object

Let $\mathbb{O}$ be the theory of object (one sort and no additional axioms). $\mathbf{Set}[\mathbb{O}] \simeq [\mathbf{Fin}, \mathbf{Set}]$.

$$\mathbf{Set}[\mathbb{O}] \models \forall xy \in U_{\mathbb{O}}.\ \neg\neg x = y.$$

The theory of rings

Let $\mathbb{R}$ be the algebraic theory of rings. $\mathbf{Set}[\mathbb{R}] \simeq \mathbf{Psh}(\mathbf{Aff}_{\text{f.p.}})$.

$$\mathbf{Set}[\mathbb{R}] \models \forall x \in U_{\mathbb{R}}.\ \neg x = 0 \to \exists y \in U_{\mathbb{R}}.\ xy = 1.$$

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For Horn theories

Theorem ([Blass, 1986])

Let $\mathbb{T}$ be a Horn (algebraic) theory. In $\mathbf{Set}[\mathbb{T}]$*, the canonical map is an equivalence,*

$$\mathrm{ev} : U_{\mathbb{T}}[\mathrm{x}] \simeq U_{\mathbb{T}}^{U_{\mathbb{T}}}.$$

Some interesting examples from this simple observation alone:

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Theorem ([Blass, 1986])

Let $\mathbb{T}$ be a Horn (algebraic) theory. In **Set**$[\mathbb{T}]$, *the canonical map is an equivalence,*

$$\mathrm{ev} : U_{\mathbb{T}}[x] \simeq U_{\mathbb{T}}^{U_{\mathbb{T}}}.$$

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$U_{\mathbb{A}}^{U_{\mathbb{A}}} \cong U_{\mathbb{A}} \times \mathbb{Z}$	every function is an affine shift of iterations

For general coherent theories

The guiding principle

Functions on generic models are all *definable*.

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Let $\mathbb{T}$ be a Horn theory. Recall in the syntactic category of $\mathbb{T}$, morphisms are given by terms in $\mathbb{T}$ modulo provable equivalence.

Theorem (Blass result reformulated)

In $\mathbf{Set}[\mathbb{T}]$*, we have*

$$U_{\mathbb{T}}^{U_{\mathbb{T}}} \cong \{\, t(\mathrm{x}) \mid t(\mathrm{x}) \text{ is a term in } \mathbb{T} \text{ with variable } \mathrm{x} \text{ and parameters from } U_{\mathbb{T}} \,\}/\sim_{\mathbb{T}}$$
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Theorem (Coh.B)

Let $\mathbb{T}$ be a coherent theory. In $\mathbf{Set}[\mathbb{T}]$, we have

$$U_{\mathbb{T}}^{U_{\mathbb{T}}} \cong \{\, \varphi(x, y) \mid \varphi(x, y) \textit{ is provably functional in } \mathbb{T} \textit{ with parameters from } U_{\mathbb{T}} \,\}/\sim_{\mathbb{T}}$$

Church's thesis in arithmetic

Let $\mathbb{T}$ be a coherent arithmetic theory (e.g. [Ye, 2026]). Let $N_{\mathbb{T}}$ be the generic $\mathbb{T}$-model.

Theorem

In $\mathbf{Set}[\mathbb{T}]$, $N_{\mathbb{T}}$ *satisfies Church's thesis with a decidable predicate* T,

$$\forall f \in N_{\mathbb{T}}^{N_{\mathbb{T}}}.\ \exists e \in N_{\mathbb{T}}.\ \forall x \in N_{\mathbb{T}}.\ \exists m \in N_{\mathbb{T}}.\ T(e, m, x, fx).$$

Proof Sketch.

By coding, there is an isomorphism $N_{\mathbb{T}} \cong \mathrm{Fml}_2(N_{\mathbb{T}})$. Let T be defined as below,

$$T(e, m, x, y) := \ulcorner m \text{ is a proof of } \varphi_e(x, y) \urcorner.$$

By (Coh.B), $N_{\mathbb{T}}^{N_{\mathbb{T}}}$ are coded by formulas, thus the desired principle holds. □

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A $U_{\mathbb{T}}$-algebra is a $\mathbb{T}$-model equipped with a map $U_{\mathbb{T}} \to A$. It has an internal *spectrum*:

$$\operatorname{Spec} A := U_{\mathbb{T}}\text{-}\mathbf{Alg}(A, U_{\mathbb{T}}).$$

Synthetic quasi-coherence for Horn theories

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Theorem (Quasi-coherence, [Blechschmidt, 2021, Blechschmidt, 2020])

If A is f.p. $U_{\mathbb{T}}$-algebra, then the canonical map is an equivalence:

$$\mathrm{ev} : A \simeq U_{\mathbb{T}}^{\operatorname{Spec} A}$$

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1. This recovers Blass: $\mathsf{Spec}\, U_{\mathbb{T}}[\mathsf{x}] \cong U_{\mathbb{T}}\text{-}\mathbf{Alg}(U_{\mathbb{T}}[\mathsf{x}], U_{\mathbb{T}}) \cong U_{\mathbb{T}}$, thus $U_{\mathbb{T}}[\mathsf{x}] \cong U_{\mathbb{T}}^{U_{\mathbb{T}}}$.

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If A is f.p. $U_{\mathbb{T}}$-algebra, then the canonical map is an equivalence:

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1. This recovers Blass's characterisation of functions:

$$\mathsf{Spec}\, U_{\mathbb{T}}[\mathrm{x}] \cong U_{\mathbb{T}}\text{-}\mathbf{Alg}(U_{\mathbb{T}}[\mathrm{x}], U_{\mathbb{T}}) \cong U_{\mathbb{T}} \quad \Rightarrow \quad U_{\mathbb{T}}[\mathrm{x}] \cong U_{\mathbb{T}}^{U_{\mathbb{T}}}.$$

2. This implies the previously mentioned curious facts. In fact, *quite powerful* for synthetic algebraic geometry [Cherubini et al., 2024], synthetic compact Hausdorff topology [Cherubini et al., 2025], synthetic domain theory [Sterling and Ye, 2025].

Synthetic quasi-coherence for Horn theories, reformulated

Following [Sterling and Ye, 2025], let $\mathcal{S}_{\mathbb{T}}$ denote the universe of sets in the internal language of $\mathbf{Set}[\mathbb{T}]$. We always have an adjunction

$$U_{\mathbb{T}}\text{-}\mathbf{Alg}^{\mathrm{op}} \underset{\mathsf{Spec}}{\overset{U_{\mathbb{T}}^{-}}{\underset{\bot}{\leftrightarrows}}} \mathcal{S}_{\mathbb{T}}$$

Theorem (Quasi-coherence for Horn theories, reformulated)

Finitely presented $U_{\mathbb{T}}$-algebras are fixed-points for the above adjunction. Equivalently, the spectrum functor is ***fully faithful*** *at finitely presented $U_{\mathbb{T}}$-algebras,*

$$\mathsf{Spec} : U_{\mathbb{T}}\text{-}\mathbf{Alg}^{\mathrm{op}}_{f.p.} \hookrightarrow \mathcal{S}_{\mathbb{T}}.$$

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Synthetic quasi-coherence for coherent theories

Let $\mathbb{T}$ be coherent, and let $U_{\mathbb{T}} \ltimes \mathbb{T}$ be the *internal* coherent theory of $\mathbb{T}$ with parameters from $U_{\mathbb{T}}$ added. $U_{\mathbb{T}} \ltimes \mathbb{T}$ classifies "$U_{\mathbb{T}}$-algebras". Let $\mathcal{C}[U_{\mathbb{T}} \ltimes \mathbb{T}]$ be its syntactic category.

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Theorem (Quasi-coherence for coherent theories)

$\mathrm{id} : U_{\mathbb{T}} \to U_{\mathbb{T}}$ *is a model of* $U_{\mathbb{T}} \ltimes \mathbb{T}$*, and it induces an* ***fully faithful*** *evaluation functor*

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To see this recovers quasi-coherence for Horn theories, let $E(\overline{\mathsf{x}})$ be a finite conjunction of equations in a Horn theory $\mathbb{T}$. We then have

$$[\![E(\overline{\mathsf{x}})]\!] \cong \{\, \overline{x} \in U_{\mathbb{T}} \mid E(\overline{x}) \,\} \cong U_{\mathbb{T}}\text{-}\mathbf{Alg}(U_{\mathbb{T}}[\overline{\mathsf{x}}]/E(\overline{\mathsf{x}}), U_{\mathbb{T}}) \cong \mathsf{Spec}\; U_{\mathbb{T}}[\overline{\mathsf{x}}]/E(\overline{\mathsf{x}}).$$

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$$\begin{array}{ccc} U_{\mathbb{T}}\text{-}\mathbf{Alg}_{f.p.}^{\mathrm{op}} & \hookrightarrow & \mathcal{C}[U_{\mathbb{T}} \ltimes \mathbb{T}] \\ & \underset{\mathsf{Spec}}{\searrow} & \downarrow [\![-]\!] \\ & & \mathcal{S}_{\mathbb{T}} \end{array}$$

Back to arithmetic

Using the quasi-coherence principle for coherent arithmetic $\mathbb{T}$, we can compare the logical structures in $\mathbf{Set}[\mathbb{T}]$ and that of synthetic computability theory [Bauer, 2006].

	arithmetic	computability
Church's thesis	functions are definable	functions are computable
dominance $\Sigma \subseteq \Omega$	definable propositions	semi-decidable propositions
enumeration $N \twoheadrightarrow \Sigma^N$	def. subsets are coded	semi-dec. sets are enumerable
fixed-point property Σ	Gödel's diaganal lemma	Kleene's recursion theorem
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Conclusion and vision I

Logic is important!

How effective the quasi-coherence principle can be applied depends on how good we understand the **syntactic properties** of logical theories.

- For algebraic theories, this means we want *normal form* theorems for polynomials:

simple	complicated
$\mathbb{O}, \mathbb{R}, \mathbb{D}, \mathbb{A}, \mathbb{B} \cdots$	$\mathbb{H}$ for Heyting algebras $\cdots$

- For coherent theories, the strongest form is *quantifier elimination.* Such theories are often *ω-categorical*, and have *atomic* classifying toposes (connection to Simpson's approach towards synthetic probability/independence!).

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What about universes?

How can we characterise Ω and $\mathcal{S}$ (or $\mathcal{S}_\infty$)? – The guiding principle should be the same.

- For Ω, [Blechschmidt, 2020] has proposed the following axiomatisation,

  $\Omega \cong \{$ geometric sentences in $U_{\mathbb{T}} \ltimes \mathbb{T}$ where disjunctions are *locally constant* $\}/\sim_{\mathbb{T}}$.

- When explicit normal forms can be computed (trivial for atomic topoi) this leads to powerful axioms: $\mathbf{Psh}(\omega)$ classifies filters on ω, and we have

$$\forall p \in \Omega.\, \exists n \in \omega \cup \{\bot\}.\, [\![n]\!] = p \quad (\omega \cup \{\bot\} \twoheadrightarrow \Omega).$$

- In other special cases this can be entangled with quasi-coherence: One can find a geometric quotient theory $\mathbb{T}$ of Heyting algebras where $U_{\mathbb{T}} \cong \Omega$!
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The end

(And please talk to me if you are interested in any of these!)

Bauer, A. (2006).
First steps in synthetic computability theory.
Electronic Notes in Theoretical Computer Science, 155:5–31.
Proceedings of the 21st Annual Conference on Mathematical Foundations of Programming Semantics (MFPS XXI).

Blass, A. (1986).
Functions on universal algebras.
Journal of Pure and Applied Algebra, 42(1):25–28.

Blechschmidt, I. (2020).
A general nullstellensatz for generalized spaces.
Unpublished note.

Blechschmidt, I. (2021).
Using the internal language of toposes in algebraic geometry.
PhD thesis, Universität Augsburg.

Cherubini, F., Coquand, T., Geerligs, F., and Moeneclaey, H. (2025).
A Foundation for Synthetic Stone Duality.
In Møgelberg, R. E. and van den Berg, B., editors, *30th International Conference on Types for Proofs and Programs (TYPES 2024)*, volume 336 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 3:1–3:20, Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

Cherubini, F., Coquand, T., and Hutzler, M. (2024).
A foundation for synthetic algebraic geometry.
Mathematical Structures in Computer Science, 34(9):1008–1053.

Orton, I. and Pitts, A. M. (2018).
Axioms for modelling cubical type theory in a topos.
Logical Methods in Computer Science, 14.

Sterling, J. and Ye, L. (2025).
Domains and classifying topoi.
arXiv preprint arXiv:2505.13096.

Ye, L. (2026).
Categorical structure in coherent theory of arithmetic.
Theoretical Computer Science, 1071:115832.