The pro-étale site for first-order logic

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Completeness means separation of syntax

Let $\mathcal{C}$ be a pretopos. Let $\mathbf{Mod}_{\mathcal{C}}(*)$ be the category of **Set**-models of $\mathcal{C}$,

$$\mathbf{Mod}_{\mathcal{C}}(*) \coloneqq \mathbf{Coh}(\mathcal{C}, \mathbf{Set}).$$

Theorem (Completeness: models *separate* syntax)

*Completeness exactly means the evaluation functor is **faithful**,*

$$\mathrm{ev}\colon \mathcal{C} \rightarrowtail [\mathbf{Mod}_{\mathcal{C}}(*), \mathbf{Set}].$$

- $\mathrm{ev}\colon \mathcal{C} \rightarrowtail [\mathbf{Mod}(\mathcal{C}), \mathbf{Set}]$ is in fact a *coherent functor*;
- There are similar embedding theorems for lex, regular, and lextensive categories ...

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Conceptual completeness means reconstruction of syntax: the lex case

Let $\mathcal{C}$ be a small lex category. Can we characterise the image $\mathcal{C} \hookrightarrow [\mathbf{Lex}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$?

Theorem (Gabriel-Ulmer)

$\mathcal{C} \hookrightarrow [\mathbf{Lex}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$ *is the full subcategory of those functors preserving:*

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- *Makkai: essential image of $\mathcal{C}$ corresponds to functors preserving ultraproducts;*
- *Lurie: we can factorise $\mathcal{C} \rightarrowtail [\mathbf{Mod}_{\mathcal{C}}(*), \mathbf{Set}]$ as follows,*

$$\mathcal{C} \xhookrightarrow{\text{よ}} \mathbf{Sh}(\mathcal{C}) \xrightarrow{\text{ev}} [\mathbf{Mod}_{\mathcal{C}}(*), \mathbf{Set}]$$

*Essential image of $\mathbf{Sh}(\mathcal{C})$ are **lax** for ultraproducts (left ultrafunctors).*

Part of the difficulty in this theorem lies in the fact that ultraproducts on $\mathbf{Mod}_{\mathcal{C}}(*)$ is **not** directly a universal property.

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2. From a quite different perspective, [Di Liberti, 2022] describes ultraproducts of models of pretopoi by **Kan extension** in the 2-category of topoi, recovering its **universal property**! Extending this idea, [Di Liberti and Ye, 2025] has provided similar characterisations for other subfragments of geometric logic, and a new conceptual completeness proof is on the way (j.w. Di Liberti & Tarantino).

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The pro-étale site

There is one important way of putting the syntax and semantics in the same context:

$$\begin{tikzcd} \mathcal{C}_{\mathrm{pro}} \arrow[r, leftrightarrow, "\simeq"] & \mathbf{Lex}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}} \\ \mathcal{C} \arrow[u, hook] & \mathbf{Mod}_{\mathcal{C}}(*)^{\mathrm{op}} \arrow[u, hook] \end{tikzcd}$$

For $F \in \mathbf{Lex}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}} \simeq \mathcal{C}_{\mathrm{pro}}$ and $\varphi \in \mathcal{C}$, the equivalence is given by $F\varphi \cong \mathcal{C}_{\mathrm{pro}}(F, \varphi)$.

- $\mathcal{C} \hookrightarrow \mathcal{C}_{\mathrm{pro}}$ is the free completion of cofiltered limits, and objects in $\mathcal{C}$ are exactly the copresentable ones; if $\mathcal{C}$ is regular (resp. lextensive), then so is $\mathcal{C}_{\mathrm{pro}}$.
- The idea is to understand the relation between between $\mathcal{C}$ and $\mathbf{Mod}_{\mathcal{C}}(*)$ via their corresponding relations with $\mathcal{C}_{\mathrm{pro}}$. In particular, we want to understand better the difference in $\mathbf{Mod}_{\mathcal{C}}(*) \hookrightarrow \mathbf{Lex}(\mathcal{C}, \mathbf{Set})$.

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When a lex functor preserves coproducts?

For simplicity, let $\mathcal{C}$ be a *lextensive* category, i.e. a lex category with disjoint and universal finite coproducts. In this case, a lex functor is a model iff it preserves *finite coproducts*.

Proposition

A lex functor $F\colon \mathcal{C} \to \mathcal{D}$ between lextensive categories is a model iff $F2 \cong 2$.

Proof.

Given $\varphi, \psi \in \mathcal{C}$, we have pullbacks,

$$\begin{tikzcd} \varphi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \varphi + \psi \arrow[d] & \psi \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ 1 \arrow[r] & 1 + 1 & 1 \arrow[l] \end{tikzcd}$$

If F preserves $1+1$, $F(\varphi+\psi)$ over $F(2) \cong 2$ again makes $F(\varphi)$, $F(\psi)$ the pullback of inclusions $1 \rightarrowtail 2$, thus by extensivity $F(\varphi+\psi) \cong F(\varphi) + F(\psi)$. □

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The profinite shape

Let $\mathcal{C}$ be lextensive. We can record the extent to which $F \in \mathcal{C}_{\mathrm{pro}} \simeq \mathbf{Lex}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$ fails to preserve coproducts by the following functor: Let $\mathbf{Prof} \coloneqq \mathbf{Fin}_{\mathrm{pro}}$ be profinite sets,

$$\begin{tikzcd} \mathcal{C}_{\mathrm{pro}} \arrow[r, "\mathcal{C}_{\mathrm{pro}}(-,2)"] \arrow[dr, "\pi"'] & \mathbf{BA}^{\mathrm{op}} \arrow[d, "\mathrm{Spec}"] \\ & \mathbf{Prof} \end{tikzcd}$$

- Given $F \in \mathcal{C}_{\mathrm{pro}} \simeq \mathbf{Lex}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$, $\mathcal{C}_{\mathrm{pro}}(F, 2) \cong F(2)$, thus F is a model iff $\pi F \cong *$.
- In fact, π is left adjoint to the constant functor $\Delta \coloneqq (\mathbf{Fin} \to \mathcal{C})_{\mathrm{pro}}$.

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Lex functors are models over its profinite connected component

Theorem

Any lex functor $F \colon \mathcal{C} \to \mathbf{Set}$ factors as a lextensive functor $\widehat{F}$ as below,

$$\begin{tikzcd} \mathcal{C} \arrow[r, dashed, "\widehat{F}"] \arrow[dr, "F"'] & \mathbf{Sh}(\pi F) \arrow[d, "\Gamma"] \\ & \mathbf{Set} \end{tikzcd}$$

Proof.

Recall $\pi F \cong \operatorname{Spec} \mathcal{C}_{\mathrm{pro}}(F, 2)$, thus a clopen $U \subseteq \pi F$ is a decidable subobject of F,

$$U \in \mathcal{C}_{\mathrm{pro}}(F, 2) \cong \mathrm{Sub}^{\mathrm{dec}}_{\mathcal{C}_{\mathrm{pro}}}(F).$$

We then construct the lift simply as follows,

$$\widehat{F}(\varphi)(U) := \mathcal{C}_{\mathrm{pro}}(U, \varphi). \qquad \square$$

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The fibration of profinite models

This leads us to look at models of $\mathcal{C}$ over profinite sets. There is a pseudo-functor

$$\mathbf{Mod}^+_{\mathcal{C}} \colon \mathbf{Prof}^{\mathrm{op}} \to \mathbf{Cat},$$

such that for any $S \in \mathbf{Prof}$, it produces the category of $\mathcal{C}$-models in $\mathbf{Sh}(S)$,

$$\mathbf{Mod}^+_{\mathcal{C}}(S) \coloneqq \mathbf{Lex}^+(\mathcal{C}, \mathbf{Sh}(S)).$$

As a **Prof**-indexed category, it also has an indexed opposite, which we denote as $\mathbf{Prof}^+_{\mathcal{C}}$,

$$\mathbf{Prof}^+_{\mathcal{C}}(S) \coloneqq \mathbf{Mod}^+_{\mathcal{C}}(S)^{\mathrm{op}}.$$

They can also be viewed as *Grothendieck fibrations* over **Prof**,

$$\mathbf{Mod}^+_{\mathcal{C}} \xrightarrow{\ \pi\ } \mathbf{Prof} \xleftarrow{\ \pi\ } \mathbf{Prof}^+_{\mathcal{C}}$$

Concretely, the total category $\mathbf{Prof}_{\mathcal{C}}^{+}$ is reminiscent of the category of *ringed spaces*:

- Objects are pairs $(S, \mathcal{O}_S)$ with $S \in \mathbf{Prof}$ and $\mathcal{O}_S \in \mathbf{Lex}^{+}(\mathcal{C}, \mathbf{Sh}(S))$;
- $(f, f^{\#}) \colon (S, \mathcal{O}_S) \to (T, \mathcal{O}_T)$ is a continuous map $f \colon S \to T$, with a map of $\mathcal{C}$-models,

$$f^{\#} : f^{*}\mathcal{O}_T \to \mathcal{O}_S.$$

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The lifting operation $\widehat{(-)}$ described before induces an equivalence over $\mathbf{Prof}$,

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Characterisation of the pro-étale site

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Connection with Diers' spectrum

Diers' work on multi-presentable categories shows the inclusion has a left multi-adjoint,

$$\mathbf{Lex}^{+}(\mathcal{C}, \mathbf{Set}) \hookrightarrow \mathbf{Lex}(\mathcal{C}, \mathbf{Set}).$$

For $F \in \mathbf{Lex}(\mathcal{C}, \mathbf{Set})$, there is a family $\{ \eta_i \colon F \to F_i \}_{i \in I}$ with $F_i \in \mathbf{Lex}^{+}(\mathcal{C}, \mathbf{Set})$, such that for any $G \in \mathbf{Lex}^{+}(\mathcal{C}, \mathbf{Set})$,

$$\mathbf{Lex}(\mathcal{C}, \mathbf{Set})(F, G) \cong \coprod_{i \in I} \mathbf{Lex}^{+}(F_i, G).$$

The index set for the multi-reflection of F is exactly πF, where the family of multi-units is given by $\{ \eta_x \colon F \cong \Gamma \widehat{F} \Rightarrow x^* \widehat{F} \}_{x \in \pi F}$,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widehat{F}"] \arrow[dr, "F"'] & \mathbf{Sh}(\pi F) \arrow[d, bend right, "\Gamma"'] \arrow[d, bend left, "x^*"] \arrow[d, phantom, "\Rightarrow"] \\
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Classify pro-étale sheaves

Importantly, $\pi\colon \mathcal{C}_{\mathrm{pro}} \to \mathbf{Prof}$ is a *fibration*! Following [van Dobben de Bruyn, 2026]:

Theorem (Exodromy for pro-étale sheaves)

For any lextensive category $\mathcal{C}$ and any category $\mathcal{E}$ with finite products,

$$\mathrm{Ex}\colon \mathbf{Sh}^+(\mathcal{C}_{\mathrm{pro}};\mathcal{E}) \simeq [\mathbf{Prof}^{\mathrm{op}},\mathbf{Cat}](\mathbf{Mod}^+_{\mathcal{C}},\mathbf{Sh}^+(\mathbf{Prof}/-;\mathcal{E})).$$

Proof.

As a fibration, we have $\mathcal{C}_{\mathrm{pro}} \simeq \int^{S\in\mathbf{Prof}} \mathbf{Prof}^+_{\mathcal{C}}(S) \times \mathbf{Prof}/S$. We may then compute,

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[\mathcal{C}^{\mathrm{op}}_{\mathrm{pro}},\mathcal{E}] &\simeq \int_{S\in\mathbf{Prof}} [\mathbf{Prof}^+_{\mathcal{C}}(S)^{\mathrm{op}} \times (\mathbf{Prof}/S)^{\mathrm{op}},\mathcal{E})\\
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Then you verify this restricts to extensive sheaves on both sides. $\square$

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The above equivalences restricts to an equivalence for étale sheaves as well,

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Proof sketch.

This relies on the fact that the essential image of the fully faithful inclusion

$$\mathbf{Sh}^{+}(\mathcal{C};\mathcal{E}) \hookrightarrow \mathbf{Sh}^{+}(\mathcal{C}_{\mathrm{pro}};\mathcal{E})$$

are those pro-étale sheaves that preserves filtered colimits, and that a morphism of stacks over **Prof** has strong implication w.r.t. preserving filtered colimits. □

The above equivalences restricts to an equivalence for étale sheaves as well,

Theorem (Exodromy for étale sheaves)

For any lextensive category $\mathcal{C}$ and locally presentable category $\mathcal{E}$,

$$\mathrm{Ex}_{\acute{e}t}\colon \mathbf{Sh}^{+}(\mathcal{C};\mathcal{E}) \simeq [\mathbf{Prof}^{\mathrm{op}},\mathbf{Cat}](\mathbf{Mod}^{+}_{\mathcal{C}},\mathbf{Sh}(-;\mathcal{E})).$$

Proof sketch.

This relies on the fact that the essential image of the fully faithful inclusion

$$\mathbf{Sh}^{+}(\mathcal{C};\mathcal{E}) \hookrightarrow \mathbf{Sh}^{+}(\mathcal{C}_{\mathrm{pro}};\mathcal{E})$$

are those pro-étale sheaves that preserves filtered colimits, and that a morphism of stacks over **Prof** has strong implication w.r.t. preserving filtered colimits. ☐

- In particular, we recover Lurie's conceptual completeness result as follows,

$$\mathbf{Sh}^+(\mathcal{C}) \simeq [\mathbf{Prof}^{\mathrm{op}}, \mathbf{Cat}](\mathbf{Mod}_{\mathcal{C}}^+, \mathbf{Sh}(-)).$$

  In fact, both $\mathbf{Mod}_{\mathcal{C}}^+$ and $\mathbf{Sh}(-)$ are *stacks* for the coherent topology on $\mathbf{Prof}$. Hence, the RHS are morphisms between two *condensed categories*!
- The same story can be used to treat the case of **pretopoi**, with more care on epis.
- Coming back to Lurie, this morally means the data of left ultrafunctors are the same as morphisms between stacks over **Prof**, which Lurie has also observed. However, this approach is slightly more flexible to adjust to lextensive categories.

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Conclusion

- In particular, we recover Lurie's conceptual completeness result as follows,

$$\mathbf{Sh}^+(\mathcal{C}) \simeq [\mathbf{Prof}^{\mathrm{op}}, \mathbf{Cat}](\mathbf{Mod}^+_{\mathcal{C}}, \mathbf{Sh}(-)).$$

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