Duality in Logic and Vector Space

Lingyuan Ye

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Tsinghua University

Outline

Dualistic Structure in Logic and Vector Space

A Practicle Example of Logic with Vector Space Models

Dualistic Structure in Logic and Vector Space

Duality in Vector Space

Recall the definition of dual vector space:

Definition

Given any vector space V over a field $\mathbb{k}$, the *dual vector space* of V, denoted as V^*, consists of all the linear functions of the type

$$\alpha : V \to \mathbb{k}$$

with linear structure defined pointwise,

$$(\alpha + \beta)(v) = \alpha(v) + \beta(v)$$

$$(k\alpha)(v) = k\alpha(v)$$

for any $\alpha, \beta \in V^*, v \in V, k \in \mathbb{k}$.

Duality in Logical Systems

- A similar dualistic structure also exists in logic – that is the relation between *semantics* and *syntax*.
- Given any formula φ in a language L, and any model $\mathcal{M}$ of this language, the usual relation between φ and $\mathcal{M}$ is the *satisfaction relation*:

$$\mathcal{M} \models \varphi \text{ or } \mathcal{M} \not\models \varphi$$

Duality in Logical Systems

- But we can also write this in a similar fashion as the case with dual vector spaces, i.e treat the model $\mathcal{M}$ as a function of the type

$$\mathcal{M} : \mathit{Fml}(L) \to \{0, 1\}$$

- Immediately, this can be generalized to many-valued logic. For a logic with W being its set of truth values, then a model $\mathcal{M}$ of this logic is a function of the type,

$$\mathcal{M} : \mathit{Fml}(L) \to W$$

Duality from a Logical Perspective

How to interpret logical duality – a particular perspective:

- In many cases, there's usually an underlying *subject* or *system* studied or described by a particular logical system:
  - a group of agents
  - a game with a set of players
  - a computer
  - a physical system
  - ...
- The dualistic parts in logic, viz. semantics and syntax, play different roles with respect to the system.

Duality from a Logical Perspective

How to interpret logical duality – a particular perspective:

- Let's give the system under consideration a name $\mathcal{S}$.
- A semantic model usually denotes *states of affairs* **of** $\mathcal{S}$:
  - epistemic states of the group of agents (Kripke models)
  - different stages of a game (game trees)
  - inner states of a computer (graphs)
  - states of a physical system (manifolds, Hilbert spaces)
  - ...
- The syntactical formulas *make assertions* **about** $\mathcal{S}$.
- Given a state (a model) $\mathcal{M}$ and a formula φ, then the value $\mathcal{M}(\varphi)$ measures the **extent** to which the state of $\mathcal{S}$ represented by $\mathcal{M}$ *commits to the assertion* made by φ.

Transmit the Logical Perspective to Vector Spaces

Duality in logic $\Rightarrow$ duality in vector space:

- A vector space $V_{\mathcal{S}}$ associated to $\mathcal{S}$ may be used to represent the *space of states* in $\mathcal{S}$;
- The dual space $V_{\mathcal{S}}^*$ is then the *space of propositions*;
- For any state $v \in V_{\mathcal{S}}$ and any proposition $\alpha \in V_{\mathcal{S}}^*$, the value $\alpha(v)$ gives the result of how a state commits to a proposition.

More importantly, this dualistic structure indicates how we may implement a particular logical system with vector space models!

Implementing Logic in Vector Space through Duality

- Consider maps E of the type

$$E : Fml(L) \to V^*$$

- A model is then a pair $\langle E, v \rangle$ for $v \in V$. The truth value is similarly given by the dual structure in V,

$$\langle E, v \rangle(\varphi) := E(\varphi)(v)$$

$\langle E, v \rangle(\varphi)$: indicates how a particular state v commits to an assertion made by φ through the map E.

Implementing Logic in Vector Space through Duality

- A natural requirement for the map E is for it to be functorial with respect to the language, viz. a map preserving the algebraic syntactical structure in L.
- This requirement ensures the interpretation of the logic into vector space to be **compositional**.

A Broader Perspective Looking at Duality

Previous ideas work in wider settings. The most general idea is the following:

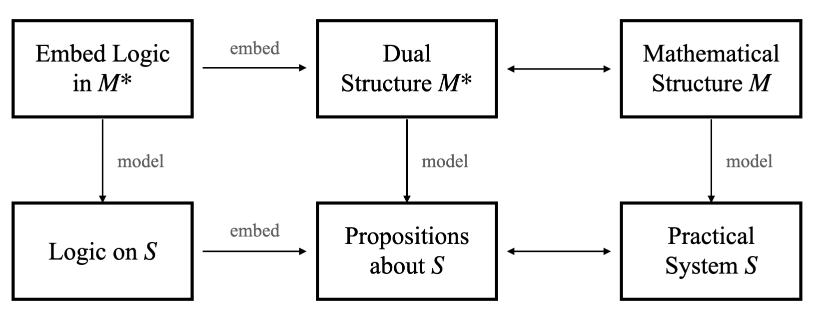


Figure 1: Duality in General

A Practicle Example of Logic with Vector Space Models

Plan

In the following, I will mainly talk about how *logic* and (general) *probability theory* can be implemented together using vector space models.

General Picture

Suppose a system $\mathcal{S}$ is under consideration. The set of *distinct states* of $\mathcal{S}$ is supposed to be finite (with cardinality n), and is denoted as $\Omega_n = \{\omega_i\}_{i \leq n}$.

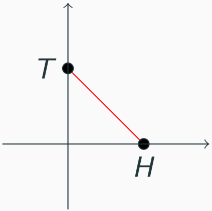
- Linear combinations $\approx$ probability mixture
- The set of all probability mixtures are the *space of states* of $\mathcal{S}$.
- The embedding function $E : Fml(L) \to V^*_{\mathcal{S}}$ will take formulas of a language to assertions about $\mathcal{S}$.
- $E(\varphi)(v)$ is the *probability* of the assertion $E(\varphi)$ to be true when $\mathcal{S}$ is at state v.

Simple Example: State Space

- A coin has two distinct states: $\Omega_{\mathcal{S}} = \{H, T\}$.
- A probability mixture on $\Omega_{\mathcal{S}}$ can be viewed as a linear combination of the following form,

$$pH + (1-p)T, \quad 0 \le p \le 1$$

- Its state space is (a subset of) $\mathbb{R}^2$:



Linear Combinations as Probability Mixtures

- Generally, a probability distribution on Ω_n is a function

$$p : \wp(\Omega_n) \to [0, 1]$$

  p respects the σ-algebra on $\wp(\Omega_n)$.

- p naturally induces a linear combination of ω_i:

$$p \longrightarrow \sum_{i=1}^{n} p(\{\omega_i\})\omega_i \in \mathrm{span}_{\mathbb{R}}\{\omega_1, \cdots, \omega_n\}$$

- $p(\{\omega_i\})$: the probability of the system being at state ω_i

Space of States

- A *standard simplex* in $\mathbb{R}^n$ is a convex set Δ^{n-1} (for it's an $(n-1)$-dimensional object in $\mathbb{R}^n$) defined to be

$$\Delta^{n-1} = \{ \ (x_1, \cdots, x_n) \mid \sum_{i=1}^{n} x_i = 1 \ \& \ \forall i. 0 \leq x_i \leq 1 \ \}$$

Fact

The probability distributions on Ω_n are in one-to-one correspondence with vectors in the standard simplex:

$$\Delta^{n-1} \subseteq \mathbb{R}^n \cong \mathrm{span}_{\mathbb{R}}\{\omega_1, \cdots, \omega_n\}$$

Simple Example: Space of Propositions

- Again for a coin, elementary propositions only consist of the following,

$$h : \text{“The coin is at heads”}$$

$$t : \text{“The coin is at tails”}$$

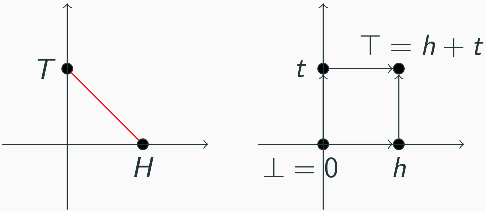
$$\perp : \text{“The coin is at heads and at tails”}$$

$$\top : \text{“The coin is at heads or at tails”}$$

  and they form a Boolean algebra.

Simple Example: Space of Propositions

- In the dual space $(\mathbb{R}^2)^*$, they are the following dual vectors,



$$t(T) = h(H) = 1, \quad t(H) = h(T) = 0$$

- The evaluation gives probability:

$$\top(pH + (1-p)T) = (h+t)(pH + (1-p)T) = p + 1 - p = 1$$

Embedding of Propositional Logic

- Generally, let $\{f_i\}$ be the dual basis of $\{\omega_i\}$
- f_i is the proposition "The system is at state ω_i"
- Elementary propositions are vertices of the unit higher cube:

$$f_{i_1} + \cdots + f_{i_r}$$

and they form a Boolean algebra P_n:

$$(f_1 + f_2) \vee (f_1 + f_3) = f_1 + f_2 + f_3$$

$$(f_1 + f_2) \wedge (f_1 + f_3) = f_1$$

$$\neg(f_1 + f_2) = f_3 + \cdots + f_n$$

$$\top = f_1 + \cdots + f_n, \ \bot = 0$$

Embedding of Propositional Logic

Now a general embedding map can be defined:

Definition

Let $\mathcal{L}_m$ be the propositional language with m propositional letters. An embedding E is defined by

1. specifying $E(p_i) \in P_n$ for each p_i.
2. recursively defining $E(\varphi \wedge \psi) = E(\varphi) \wedge E(\psi), \cdots$

Evaluation gives probability:

$E(\varphi)(v) =$ the probability of $E(\varphi)$ being true given distribution v

Embedding of Propositional Logic

There's an alternative way to think about the embedding, through the following facts:

Fact

1. Let B_n denote the set of subspaces spanned by subsets of $f_1, \cdots, f_n$. B_n is closed under intersection, sum and complements of subspaces.
2. B_n forms a Boolean algebra under those operations.
3. B_n as a Boolean algebra is isomorphic to P_n.

Further Directions to Explore

- Extend the set of "propositions" you want to consider about
  - preference, vague propositions, $\cdots \Rightarrow$ Hannes Leitgeb's work
- Explore *(de)composition* of systems
  - FOL; how logic behave with (de)composition of systems
- Explore *transition* of states
  - adding dynamic aspect into the picture
- Outside classical framework: General probability theory
- There's also no need to stick to probability theory!

The End

Thank you!

Q & A