Incompleteness and Intensionality via Universal Property of Arithmetic

Lingyuan Ye

2024.11.16, PSSL, Leiden

University of Cambridge

Introduction

"Violations" of (GII)

Identify a theory as a set of axioms. To formalise PA in itself, we need a formula $\mathrm{ax}(x)$ to enumerate all PA axioms.

Example ([Feferman, 1960])

There is a formula $\mathrm{ax}(x)$ representing the PA axioms,

$$\begin{cases} \varphi \in \mathrm{PA} & \mathrm{PA} \vdash \mathrm{ax}(\ulcorner \varphi \urcorner) \\ \varphi \notin \mathrm{PA} & \mathrm{PA} \vdash \neg \mathrm{ax}(\ulcorner \varphi \urcorner) \end{cases}$$

However, with respect to which,

$$\mathrm{PA} \vdash \mathrm{Con}_{\mathrm{ax}} := \neg \exists x. \mathrm{Prf}_{\mathrm{ax}}(x, \ulcorner \bot \urcorner).$$

"Violations" of (GII)

Identify a theory as a set of axioms. To formalise PA in itself, we need a formula $\mathrm{ax}(x)$ to enumerate all PA axioms.

Example ([Feferman, 1960])

There is a formula $\mathrm{ax}(x)$ representing the PA axioms,

$$\begin{cases} \varphi \in \mathrm{PA} & \mathrm{PA} \vdash \mathrm{ax}(\ulcorner\varphi\urcorner) \\ \varphi \notin \mathrm{PA} & \mathrm{PA} \vdash \neg\mathrm{ax}(\ulcorner\varphi\urcorner) \end{cases}$$

However, with respect to which,

$$\mathrm{PA} \vdash \mathrm{Con}_{\mathrm{ax}} := \neg\exists x.\mathrm{Prf}_{\mathrm{ax}}(x, \ulcorner\bot\urcorner).$$

Externally equivalent notions might be internally different in PA, because weaker meta-theories identify less notions.

Central Question for Intensional Aspects of (GII)

How to identify the correct copy of PA in weaker meta-theories?

One way to do this: via **universal properties**.

A Mathematical Perspective

Externally equivalent notions might be internally different in PA, because weaker meta-theories identify less notions.

Central Question for Intensional Aspects of (GII)

How to identify the correct copy of PA in weaker meta-theories?

One way to do this: via **universal properties**.

Externally equivalent notions might be internally different in PA, because weaker meta-theories identify less notions.

Central Question for Intensional Aspects of (GII)

How to identify the correct copy of PA in weaker meta-theories?

One way to do this: via **universal properties**.

Universal Property of Arithmetic

Very briefly, categorical logic is the following equivalence,

$$\mathcal{C}[-] : \{F\text{-Theories}\} / \text{equiv.} \simeq \{F\text{-Categories}\}$$

Examples of F: Coherent, Heyting, Boolean, ...

- $\mathcal{C}[T]$ is the category of *T-classes*.
- Formalising mathematics in $T \simeq$ working internally in $\mathcal{C}[T]$.

Theorem

$\mathcal{C}[\text{PA}]$ *is the initial Boolean cat. with parametrised NNO.*

Similar results for CA, the Σ_1-fragment of $I\Sigma_1$ w.r.t. coherent cat., and for HA w.r.t. Heyting cat.; cf. [Ye, 2023].

Universal Property of Arithmetic

Very briefly, categorical logic is the following equivalence,

$$\mathcal{C}[-] : \{F\text{-Theories}\}/\text{equiv.} \simeq \{F\text{-Categories}\}$$

Examples of F: Coherent, Heyting, Boolean, ...

- $\mathcal{C}[T]$ is the category of T*-classes*.
- Formalising mathematics in $T \simeq$ working internally in $\mathcal{C}[T]$.

Theorem

$\mathcal{C}[\mathrm{PA}]$ *is the initial Boolean cat. with parametrised NNO.*

Similar results for CA, the Σ_1-fragment of $I\Sigma_1$ w.r.t. coherent cat., and for HA w.r.t. Heyting cat.; cf. [Ye, 2023].

Universal Property of Arithmetic

Very briefly, categorical logic is the following equivalence,

$$\mathcal{C}[-] : \{F\text{-Theories}\}\,/\text{equiv.} \simeq \{F\text{-Categories}\}$$

Examples of F: Coherent, Heyting, Boolean, ...

- $\mathcal{C}[T]$ is the category of *T-classes.*
- Formalising mathematics in $T \simeq$ working internally in $\mathcal{C}[T]$.

Theorem

$\mathcal{C}[\mathrm{PA}]$ *is the initial Boolean cat. with parametrised NNO.*

Similar results for CA, the Σ_1-fragment of $I\Sigma_1$ w.r.t. coherent cat., and for HA w.r.t. Heyting cat.; cf. [Ye, 2023].

Abstract Categorical Framework

The Set Up of Categorical Incompleteness

Let $\mathcal{L}$ be some family of left exact cat. (specified algebraically):

- Suppose $\mathcal{C}$ is the initial object in $\mathcal{L}$;
- And $\mathbb{C}$ is the initial object in $\mathcal{L}_{\mathcal{C}}$.

The coding functor is induced by initiality of $\mathcal{C}$,

$$\ulcorner - \urcorner : \mathcal{C} \to \mathrm{Ext}(\mathbb{C}) := \mathcal{C}(1, \mathbb{C}) \in \mathcal{L}.$$

Provability is formalised via internal global section,

$$\Gamma : \mathrm{Ext}(\mathbb{C}) \to \mathcal{C}, \quad \alpha \mapsto \begin{tikzcd} \Gamma(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{C}_1 \arrow[d, "\langle s,t \rangle"] \\ 1 \arrow[r, "\langle \ulcorner 1 \urcorner, \alpha \rangle"'] & \mathbb{C}_0 \times \mathbb{C}_0 \end{tikzcd}$$

The composite $\Box = \Gamma \circ \ulcorner - \urcorner$ is the provability functor.

The Set Up of Categorical Incompleteness

Let $\mathcal{L}$ be some family of left exact cat. (specified algebraically):

- Suppose $\mathcal{C}$ is the initial object in $\mathcal{L}$;
- And $\mathbb{C}$ is the initial object in $\mathcal{L}_{\mathcal{C}}$.

The coding functor is induced by initiality of $\mathcal{C}$,

$$\ulcorner - \urcorner : \mathcal{C} \to \mathrm{Ext}(\mathbb{C}) := \mathcal{C}(1, \mathbb{C}) \in \mathcal{L}.$$

Provability is formalised via internal global section,

$$\Gamma : \mathrm{Ext}(\mathbb{C}) \to \mathcal{C}, \quad \alpha \mapsto \begin{tikzcd} \Gamma(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{C}_1 \arrow[d, "\langle s,t \rangle"] \\ 1 \arrow[r, "\langle \ulcorner 1 \urcorner, \alpha \rangle"'] & \mathbb{C}_0 \times \mathbb{C}_0 \end{tikzcd}$$

The composite $\Box = \Gamma \circ \ulcorner - \urcorner$ is the provability functor.

The Set Up of Categorical Incompleteness

Let $\mathcal{L}$ be some family of left exact cat. (specified algebraically):

- Suppose $\mathcal{C}$ is the initial object in $\mathcal{L}$;
- And $\mathbb{C}$ is the initial object in $\mathcal{L}_{\mathcal{C}}$.

The coding functor is induced by initiality of $\mathcal{C}$,

$$\ulcorner - \urcorner : \mathcal{C} \to \mathrm{Ext}(\mathbb{C}) := \mathcal{C}(1, \mathbb{C}) \in \mathcal{L}.$$

Provability is formalised via internal global section,

$$\Gamma : \mathrm{Ext}(\mathbb{C}) \to \mathcal{C}, \quad \alpha \mapsto \begin{tikzcd} \Gamma(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{C}_1 \arrow[d, "\langle s,t \rangle"] \\ 1 \arrow[r, "\langle \ulcorner 1 \urcorner, \alpha \rangle"'] & \mathbb{C}_0 \times \mathbb{C}_0 \end{tikzcd}$$

The composite $\Box = \Gamma \circ \ulcorner - \urcorner$ is the provability functor.

The Set Up of Categorical Incompleteness

Let $\mathcal{L}$ be some family of left exact cat. (specified algebraically):

- Suppose $\mathcal{C}$ is the initial object in $\mathcal{L}$;
- And $\mathbb{C}$ is the initial object in $\mathcal{L}_{\mathcal{C}}$.

The coding functor is induced by initiality of $\mathcal{C}$,

$$\ulcorner - \urcorner : \mathcal{C} \to \mathrm{Ext}(\mathbb{C}) := \mathcal{C}(1, \mathbb{C}) \in \mathcal{L}.$$

Provability is formalised via internal global section,

$$\Gamma : \mathrm{Ext}(\mathbb{C}) \to \mathcal{C}, \quad \alpha \mapsto \begin{tikzcd} \Gamma(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{C}_1 \arrow[d, "\langle s,t \rangle"] \\ 1 \arrow[r, "\langle \ulcorner 1 \urcorner, \alpha \rangle"'] & \mathbb{C}_0 \times \mathbb{C}_0 \end{tikzcd}$$

The composite $\Box = \Gamma \circ \ulcorner - \urcorner$ is the provability functor.

We say $\mathcal{L}$ is constructive if $\mathcal{L}$ is closed under Artin glueing.

Theorem

If $\mathcal{L}$ is constructive, then $\Box$ satisfies the following conditions:

(P) *$\Box$ is a pointed functor $\eta : 1_{\mathcal{C}} \Rightarrow \Box$;*

(L) *It has the Löb's property: $\Box 1 = 1$ is the initial $\Box$-algebra,*

$$\begin{tikzcd}
\Box 1 \arrow[r, dashed, "\Box\omega"] \arrow[d] & \Box\varphi \arrow[d] \\
1 \arrow[r, dashed, "\exists!\omega"'] & \varphi
\end{tikzcd}$$

About the Proof

The Constructive Case

- The above structure only requires $\mathbb{C} \in \mathcal{L}_{\mathcal{C}}$.
- Joyal studied list pretopoi (cf. [van Dijk and Oldenziel, 2020]). The proof uses initiality of $\mathcal{C}$, e.g. 1 is projective.
- However, there are much more general proofs which works with any $(\mathcal{C}, \mathbb{C}, 1 \Rightarrow \square)$; see [Ramesh, 2023].

Theorem (Non-Constructive Case [Feferman, 1960])

Let $\mathcal{C}[\mathrm{PA}]_{\Sigma}$ be the smallest coherent full subcategory containing the PNNO. If $\mathbb{C} \in \mathcal{C}[\mathrm{PA}]_{\Sigma}$, then (GII) holds.

About the Proof

The Constructive Case

- The above structure only requires $\mathbb{C} \in \mathcal{L}_{\mathcal{C}}$.
- Joyal studied list pretopoi (cf. [van Dijk and Oldenziel, 2020]). The proof uses initiality of $\mathcal{C}$, e.g. 1 is projective.
- However, there are much more general proofs which works with any $(\mathcal{C}, \mathbb{C}, 1 \Rightarrow \Box)$; see [Ramesh, 2023].

Theorem (Non-Constructive Case [Feferman, 1960])

Let $\mathcal{C}[\mathrm{PA}]_{\Sigma}$ be the smallest coherent full subcategory containing the PNNO. If $\mathbb{C} \in \mathcal{C}[\mathrm{PA}]_{\Sigma}$, then (GII) holds.

Outlooks

For HA

The existence of $1 \Rightarrow \Box$ for HA is surprising:

- The usual formalisation of HA does not seem to satisfy (P).
- There are stronger axiomatisations that does [Visser, 1982].
- What is the "correct" provability logic of HA?

For PA

Löb's theorem for PA seems to suggest $(\mathcal{C}[\mathrm{PA}], \Box)$ also satisfies (L).

Is there is nice categorical proof of this?

Outlooks

For HA

The existence of $1 \Rightarrow \Box$ for HA is surprising:

- The usual formalisation of HA does not seem to satisfy (P).
- There are stronger axiomatisations that does [Visser, 1982].
- What is the "correct" provability logic of HA?

For PA

Löb's theorem for PA seems to suggest $(\mathcal{C}[\mathrm{PA}], \Box)$ also satisfies (L). Is there is nice categorical proof of this?

References i

Feferman, S. (1960).
Arithmetization of metamathematics in a general setting.
Fundamenta Mathematicae, 49:35–92.

Ramesh, S. (2023).
***Introspective Theories and Geminal Categories*.**
PhD thesis, University of California, Berkeley.

van Dijk, J. and Oldenziel, A. G. (2020).
Gödel incompleteness through arithmetic universes after a. joyal.
arXiv preprint arXiv:2004.10482.

Visser, A. (1982).
On the completenes principle: A study of provability in heyting's arithmetic and extensions.
Annals of Mathematical Logic, 22(3):263–295.

Ye, L. (2023).
Categorical structure in theory of arithmetic.
arXiv preprint arXiv:2304.05477.