Diagonalisation and Well-founded Naming

Lingyuan Ye, joint work with B. Grabmayr and V. Halbach

2023.09.19, Oxford

ILLC, University of Amsterdam

Let $\mathcal{L}_0$ be the language of *PA*; let $\mathcal{L}$ be an extension of $\mathcal{L}_0$, with an additional function symbol for each p.r. function. Let Σ be an $\mathcal{L}$-theory containing R and all true identities between closed terms.

Let Fml_x and Term_x be the set of formulas and terms with at most x free. $\mathrm{Sent} \subset \mathrm{Fml}_x$ and $\mathrm{CTerm} \subset \mathrm{Term}_x$ are closed expressions.

A *naming function* $\ulcorner - \urcorner$ is the composition of two functions $\nu \circ \alpha$:

- α is a *numbering* function providing codes for expressions;
- ν is a *numeral* function mapping numbers to closed terms.

Preliminaries

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Introduction

In Search of Self-Referential Sentences

Self-referential sentence: “*the*” sentence asserting it has property P.

- *Gödel sentence*: “the” sentence expresses itself is unprovable.
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- numbering and numeral
- how the property is expressed
- diagonalisation methods
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Following [Heck, 2007] & [Halbach and Visser, 2014a], we consider an extensional criterion for self-reference via closed terms:

Definition (Kreisel-Henkin Criterion)

The fixed points of $\varphi(x)$ is of the form $\varphi(t)$, with a closed term t s.t. $\Sigma \vdash t = \ulcorner \varphi(t) \urcorner$.

Diagonalisation Matters

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A *refutable* diagonal sentence of provability:

Example [Kreisel, 1953]

- Diagonalise $x \neq x \land \mathrm{Bew}(x)$ s.t. $\Sigma \vdash t = \ulcorner t \neq t \land \mathrm{Bew}(t) \urcorner$.
- Identify new provability predicate $\mathrm{Bew}_K(x) :\equiv x \neq t \land \mathrm{Bew}(x)$.
- $\Sigma \vdash \neg \mathrm{Bew}_K(t)$, but $\Sigma \vdash \mathrm{Bew}_K(s)$, if s is standardly obtained.

Intuitively, t "happens" to be a fixed point of $\mathrm{Bew}_K(x)$.

We can generalise:[1]

Example (Generalised Kreisel)

For $\varphi(x)$ with n (marked) occurence of x,

- First find some fixed point t with $\Sigma \vdash t = \ulcorner \varphi(t) \urcorner$.
- Construct $\{\varphi^t_S\}_{S \in \wp(n)/n}$ indexed by non-full subsets of n, s.t. $\varphi^t_S \equiv \varphi[x_{(i)}/t]_{i \in S}$, where $x_{(i)}$ is the i-th occurence of x:

$$\varphi(\cdots, x_{(i)}, \cdots) \mapsto \varphi^t_S \equiv \varphi(\cdots, t, \cdots)$$

- By definition, $\varphi^t_S(t) \equiv \varphi(t)$, thus $\Sigma \vdash t = \ulcorner \varphi^t_S(t) \urcorner$.

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Goal

In this talk:

- Make precise the notion of "*(non-)accidental* fixed points" via a precisely definition of *uniform diagonalisation*.
- Under *well-motivated constraint for naming*, exclude Kreisel's construction and alike.

Uniform Diagonalisation

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Definition (Basic Uniform Operators)

- Substitution:

$$\mathrm{Sub}_f : \mathrm{Fml}_x \times \mathrm{Term}_x \to \mathrm{Fml}_x, \quad \mathrm{Sub}_t : \mathrm{Term}_x \times \mathrm{Term}_x \to \mathrm{Term}_x.$$

- Naming functions:[2]

$$\ulcorner - \urcorner_f : \mathrm{Fml}_x \to \mathrm{Term}_x, \quad \ulcorner - \urcorner_t : \mathrm{Term}_x \to \mathrm{Term}_x.$$

[2] Notice that these two functions depend on the choice of naming.

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Definition (Basic Uniform Operators – Cont.)

- For an n-ary connective $\star$,

$$\overline{\star} : \mathrm{Fml}_x^n \to \mathrm{Fml}_x.$$

  Similarly for function symbols and relation symbols.

- Finally, we are allowed to use the designated free variable,

$$\overline{x} : 1 \to \mathrm{Term}_x.$$

Uniform Diagonalisation — Cont.

Definition

- d is *uniform* if it can be obtained via composition and Cartesian product structure with *basic uniform operators* and *projections*:

$$\begin{cases} f : A \to B \in \mathsf{U} \\ g : B \to C \in \mathsf{U} \end{cases} \Rightarrow g \circ f : A \to C \in \mathsf{U}.$$

$$\begin{cases} f : A \to B \in \mathsf{U} \\ g : A \to C \in \mathsf{U} \end{cases} \Rightarrow \langle f, g \rangle : A \to B \times C \in \mathsf{U}.$$

- $d : \mathrm{Fml}_x \to \mathrm{Term}_x$ is a *uniform diagonal operator* if $d \in \mathsf{U}$ and

$$\operatorname{img} d \subseteq \mathrm{CTerm}\ (d \text{ is closed})\ \ \&\ \Sigma \vdash d\varphi = \ulcorner \varphi(d\varphi) \urcorner.$$

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Remark

- All basic operators and projections are intuitively *uniform*.
- The definition of uniformity is recursive.
- A syntax theory contains *substitution*, *quotation*, and *concatenation*: these are exactly our basic constituents.
- A *typed* approach to always yield well-formed expressions.
- Uni. dia. depends on the chosen naming and the theory.

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Example I: Gödel's Diagonal Operator

Let sub_G be function symbol in the language satisfying

$$\Sigma \vdash \mathrm{sub}_G(\ulcorner \varphi(x) \urcorner, \ulcorner e \urcorner) = \ulcorner \varphi(\ulcorner e \urcorner) \urcorner.$$

Let $d_G : \mathrm{Fml}_x \to \mathrm{Term}_x$ be the closed operator as follows,

$$d_G \varphi \equiv \mathrm{sub}_G(\ulcorner \varphi(\mathrm{sub}_G(x, x)) \urcorner, \ulcorner \varphi(\mathrm{sub}_G(x, x)) \urcorner).$$

Proposition

d_G is a uniform diagonal operator.

Example I: Gödel's Diagonal Operator — Cont.

Proof.

Picking out the term $\mathrm{sub}_G(x, x)$ is uniform:

$$1 \xrightarrow{\langle \overline{x}, \overline{x} \rangle} \mathrm{Term}_x \times \mathrm{Term}_x \xrightarrow{\overline{\mathrm{sub}_G}} \mathrm{Term}_x$$

Gödel's construction can be viewed as the following composition,

$$\begin{tikzcd}
\mathrm{Fml}_x \arrow[r, "\langle \mathrm{id}, \overline{\mathrm{sub}_G(x,x)} \rangle"] & \mathrm{Fml}_x \times \mathrm{Term}_x \arrow[r, "\mathrm{Sub}_f"] & \mathrm{Fml}_x \arrow[d, "\ulcorner - \urcorner_f"] \\
\mathrm{Term}_x & \mathrm{Term}_x \times \mathrm{Term}_x \arrow[l, "\overline{\mathrm{sub}_G}"'] & \mathrm{Term}_x \arrow[l, "\langle \mathrm{id}, \mathrm{id} \rangle"']
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$\square$

Example II: Functional Diagonalisation

Fix some unary function symbol $\underset{.}{d}$. Let D be a p.r. function satisfying

$$D(\#\varphi(x)) = \#\varphi(\underset{.}{d}(\ulcorner\varphi\urcorner)).$$

Choose a *theory* Σ_D such that $\underset{.}{d}$ is interpreted as D,

$$\Sigma_D \vdash \underset{.}{d}(\ulcorner\varphi\urcorner) = \ulcorner\varphi(\underset{.}{d}(\ulcorner\varphi\urcorner))\urcorner.$$

$d_D(\varphi) \equiv \underset{.}{d}(\ulcorner\varphi\urcorner)$ is evidently uniform, and uni. dia. for Σ_D.

Uniformity Enough?

Now the natural question is this:

Question

Is there is a uni. dia. d and a $\mathrm{Bew}^\star(x)$ that

$$\mathrm{Bew}^\star(x) \equiv x \neq d(\mathrm{Bew}^\star(x)) \wedge \mathrm{Bew}(x)$$

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A Counterexample

Work in some Σ_D. Let $\mathrm{Bew}^\star(x)$ be $\mathrm{Bew}^\star(x) \equiv x \neq \underset{\cdot}{d}(0) \wedge \mathrm{Bew}(x)$.

Define a new numbering α (assume $\#$ is positive) s.t.

$$\alpha(\chi) = \begin{cases} 0 & \chi \equiv \mathrm{Bew}^\star(x) \\ \#\chi & \text{otherwise} \end{cases}$$

Now $d_D(\mathrm{Bew}^\star(x)) \equiv \underset{\cdot}{d}(0)$, hence evidently

$$\mathrm{Bew}^\star(x) \equiv x \neq d_D(\mathrm{Bew}^*(x)) \wedge \mathrm{Bew}(x).$$

The actual construction uses Kleene's *Primitive Recursion Theorem*.

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Diagonalisation and Naming

What is Really Going On?

If we look closely at the formula we have obtained before,

$$\mathsf{Bew}^{\star}(x) \equiv x \neq \underset{\cdot}{d}(0) \wedge \mathsf{Bew}(x),$$

since $\alpha(\mathsf{Bew}^{\star}(x)) = 0$, it contains *its own α-name*.[2]

This points at certain *circular* features in the set up of meta-logic:

- If we read a sentence by the usual way of unquoting all the names in it, this procedure will never terminate for $\mathsf{Bew}^{\star}(x)$.

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Naming Relation

Let $\preceq$ denote the subexpression relation. For a naming function $\ulcorner - \urcorner$,

Definition

For two $\mathcal{L}$-expressions e, e',

$$e \lhd e' \Leftrightarrow \exists e''. e \preceq e'' \,\&\, \ulcorner e'' \urcorner \preceq e'.$$

$\lhd$ is the *weak naming relation* for $\ulcorner - \urcorner$. If we want to stress α, ν, we also write $\lhd^{\alpha,\nu}$. $\lhd_*$ is the transitive closure of $\lhd$.

In the previous example, the naming relation $\lhd$ has a *loop*.

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Uniform Diagonalisation and Naming

Proposition (φ must be named in its diagonal term)

If d is uni. dia., then for any $\varphi(x)$, there is a term s that

$$\varphi(s) \lhd_* d\varphi.$$

Key Point.

Uniformity forces you to use $\ulcorner - \urcorner_f$ at some point. □

Theorem (No Uniform Kreisel Construction for Non-circular $\lhd_*$)

For irreflexive $\lhd_*$ and uni. dia. d, no $\varphi(x)$ can contain $d\varphi$.

Proof.

If $d\varphi \preceq \varphi(x)$, then $d\varphi \preceq \varphi(s)$ and $\varphi(s) \lhd_* d\varphi$, hence $d\varphi \lhd_* d\varphi$. □

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Well-Founded Naming

Naming as Quotation

Natural question is: How to justify irreflexivity of $\lhd_*$?

In philosophy, the naming $\ulcorner - \urcorner$ is often thought of as the arithmetic proxy of quotation, e.g. when explaining the "disquotation" schema

$$\mathsf{T}(\ulcorner\varphi\urcorner) \leftrightarrow \varphi \quad \Leftrightarrow \quad \text{'S' is true iff S.}$$

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Every naming function which resembles quotation induces a well-founded naming relation.

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We studied string theoretic *literal quotation*, which preserves “literal information”. Including:

- Usual standard naming for arithmetic;
- Structural descriptive names [Tarski, 1936];
- Naming devices for syntax theories [Halbach and Leigh, 2021];
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Under various mild conditions, we showed *every literal naming function induces a well-founded naming relation*:

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- Our developments also suggest more clearly how different dimension of self-reference can interplay intimately;
- Much more applications and examples in our paper.

Thanks for Listening! Q&A

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References i

Halbach, V. and Leigh, G. (2021).
The Road to Paradox: A Guide to Syntax, Truth, and Modality.
Cambridge University Press.
To be published.

Halbach, V. and Visser, A. (2014a).
Self-reference in arithmetic I.
Review of Symbolic Logic, 7(4):671–691.

Halbach, V. and Visser, A. (2014b).
Self-reference in arithmetic II.
Review of Symbolic Logic, 7(4):692–712.

Heck, R. K. (2007).
Self-reference and the languages of arithmetic.
Philosophia Mathematica, 15(1):1–29.
(originally published under the name ”Richard G. Heck, Jr”).

Kreisel, G. (1953).
On a problem of Henkin’s.
Indagationes Mathematicae, (15):405–406.

Tarski, A. (1936).
Der Wahrheitsbegriff in den formalisierten Sprachen.
Studia Philosophica, 1:261–405.
Reprinted as ‘The Concept of Truth in Formalized Languages’ in [Tarski, 1956], pp. 152–278. Page references are given for the translation.

Tarski, A. (1956).
Logic, Semantics, Metamathematics.
Oxford, Clarendon Press.

Appendix I: Ill-Founded, Non-Circular Kreisel Construction

Proposition

There exists a theory Σ, a numbering α and numeral ν with ill-founded but non-circular $\lhd_*^{\alpha,\nu}$, and a family $\mathrm{Bew}_n(x)$, satisfying

- $\mathrm{Bew}_n(x)$ weakly represents provability of Σ w.r.t. α;
- $\Sigma \vdash \neg\mathrm{Bew}_n(t_n)$;
- t_n does not occur in $\mathrm{Bew}_n(x)$, but in $\mathrm{Bew}_{n-1}(x)$;

where $t_n = d(\mathrm{Bew}_n(x))$ for some uniform diagonal operator.

This suggests a chain

$$\cdots \lhd_*^{\alpha,\nu} t_{n+1} \lhd_*^{\alpha,\nu} t_n \lhd_*^{\alpha,\nu} t_{n-1} \lhd_*^{\alpha,\nu} \cdots \lhd_*^{\alpha,\nu} t_1$$

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Appendix II: String Theoretic Literal Quotation

Let $\mathcal{A}$ be an alphabet. *Literal function* $L : (\mathcal{A} \cup \{\epsilon\}) \times \mathcal{A}^* \to \mathcal{A}^* \setminus \{\epsilon\}$.

Let G satisfy $e, f \preceq G(e, f)$, and e, f don't overlap in $G(e, f)$.

A *prenaming function* N' is recursively defined:

$$\begin{cases} N'(\epsilon) = L(\epsilon, \epsilon) \\ N'(a_1 \cdots a_n) = G(N'(a_{\pi_s(1)} \cdots a_{\pi_s(n-1)}), L(a_{\pi_s(n)}, s)) \end{cases}$$

where $s = a_1 \cdots a_n$.

A *literal naming function*: $N(e) = B(e)N'(e)E(e)$.

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Appendix III: Literal Naming Function for Arithmetic

Let $\#$ be a monotone numbering. $\ulcorner - \urcorner^{\#}$ is a literal naming function:

$$L(a, e) = \begin{cases} \ulcorner a \urcorner^{\#} & \mathrm{lh}(e) \leq 1 \\ \underbrace{\mathsf{S} \cdots \mathsf{S}}_{\Delta \text{ times}} & e = sa \,\&\, \Delta = \#(sa) - \#(a) \end{cases}$$

The naming function then satisfy

$$\ulcorner e \urcorner^{\#} = \begin{cases} \ulcorner e \urcorner^{\#} & \mathrm{lh}(e) \leq 1 \\ L(a_n, a_1 \cdots a_n) \ulcorner a_1 \cdots a_{n-1} \urcorner^{\#} & e = a_1 \cdots a_n \end{cases}$$

In particular, $\lhd^{\#}$ will be *well-founded*.

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