Unification of Semantics of Modal Logic via Topological Categories [a]

LIRa Seminar

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2022.10.06

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[a]This work is derived from my bachelor thesis, supervised by Prof. van Benthem.

First Words

What *not* to expect from this presentation:

- Unification as in subsuming every important result.
- Unification as in covering every interesting branches.
- Unification not foundamentally biased by my knowledge.

Second Words

What this presentation *is* about:

- Unification as in explaning the **common mathematical structure** behind various types of semantics of model logic.
- Unification as in finding an **exact correspondence** between different *syntactic* features of (various extensions of) modal logic and the *semantic* structure behind.
- Unification as in **generalising** local developments within a specific area/topic to find its global applications.

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Structure of this Talk

Motivation and Introduction

Background

This work initially is an attempt to further develop the idea of "*Information Landscape*" introduced by van Benthem [11, 10]:

$$\begin{tikzcd}
\text{coarser} \arrow[ddd, Rightarrow] & \text{bare set of possibilities/epistemic models} \arrow[d, dashed, shift right, "\text{embed}"'] \\
& \text{preordered models} \arrow[u, shift right, "\text{generate}"'] \arrow[d, dashed, shift right, "\text{embed}"'] \\
& \text{topological models} \arrow[u, shift right, "\text{generate}"'] \arrow[d, dashed, shift right, "\text{embed}"'] \\
\text{finer} & \text{evidence models} \arrow[u, shift right, "\text{generate}"'] \\
{} \arrow[rr, Rightarrow, "\text{dynamic}"'] & & {}
\end{tikzcd}$$

Figure 1: A Corner of the Landscape of Information

Problems as Hand

The initial consideration driven this work:

- *Justification* for putting them into a single diagramme?
- *Mathematical* description for the vertical connections between these different levels?
- *Interaction* between the vertical transformations and the horizontal dynamics?

Our work: Provide a *single* theoretic framework based on *topological categories* to address them all, and find new things.

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Our work: Provide a *single* theoretic framework based on *topological categories* to address them all, and find new things.

Why Topological Categories

Why *categories*?

- Category theory is a nice tool to study *structural* properties of mathematical objects (the use of it is already suggested in [11]).

Why *topological categories*?

- Topological categories (over **Set**) are formalising the category of "sets equipped with *additional geometric data*."

Cornerstone: The familiar semantics (relational, topological, neighbourhood, algebraic, and their various subclasses) are all instances of topological categories (over **Set**).

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Mathematical Background

Category of Structures

The idea (perhaps dated to Bourbaki school [4]) that *structures* over some category should be described as a *forgetful functor*,

$$|-| : \mathcal{A} \to \mathbf{Set}.$$

[1] further axiomatised when such structure is of an *algebraic* or of a *geometric* type — a property of the *functor*, not of the total category:

$$|-| : \mathbf{TopGrp} \to \mathbf{Grp}, \quad |-| : \mathbf{TopGrp} \to \mathbf{Top}.$$

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Topological Categories as Bifibrations

Any such forgetful functor should be *faithful*, meaning morphisms from A to B in $\mathcal{A}$ forms a subset of $\mathbf{Set}(|A|, |B|)$.

Facts and terminologies:

- $f : |A| \to |B|$ belongs to $\mathcal{A} \iff f$ is an $\mathcal{A}$-*morphism*.
- The *fibre* $\mathcal{A}_X$ over X is a preorder that,

$$\mathcal{A}_X := \{ A \in \mathcal{A} \mid |A| = X \}, \quad A \leq B \iff \mathrm{id}_X : |A| \to |B| \in \mathcal{A}.$$

- $\mathcal{A}$ is *fibre-small* if each fibre is a *set* (rather than proper class).

Topological categories are funtors satisfying certain very strong lifting properties, which enables many geometric construction.

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Here we record a theorem[1] (or in our case, a definition):

Theorem (Topological Categories as Indexed Categories)

A (fibre-small) topological category is equivalent to a functor

$$\mathcal{A}_- : \mathbf{Set}^{\mathrm{op}} \to \mathbf{InfL}, \quad \text{or} \quad \mathcal{A}_- : \mathbf{Set} \to \mathbf{SupL}.$$

- Each fibre is a complete lattice.
- For any function $f : X \to Y$, we have an adjunction

$$f_! : \mathcal{A}_X \rightleftarrows \mathcal{A}_Y : f^*.$$

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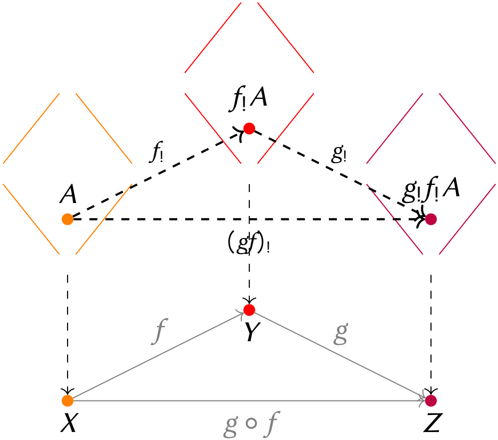
The generic pictures are as follows:



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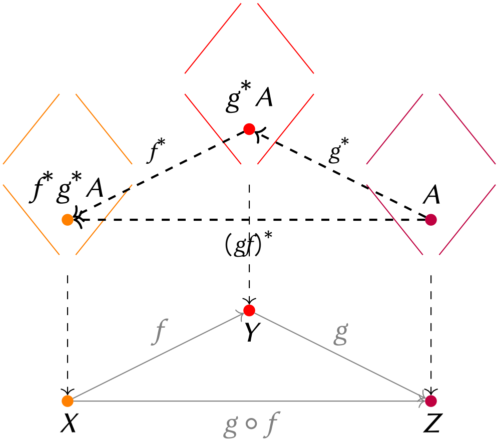

The geometric meaning behind such structures:

- The top and bottom element corresponds to the *indiscrete* and *discrete* space over a set, respectively.
- Pullbacks are describing the *finest* space in the domain making the underlying function continuous.
- Pushforwards are describing the *coarsest* space in the codomain making the underlying function continuous.
- It means given any function $f\colon |A| \to |B|$,

$$f \text{ is an } \mathcal{A}\text{-morphism } \Leftrightarrow f_! A \le B \Leftrightarrow A \le f^* B.$$

Unification of Semantics

Semantic Categories Are Topological

Main examples of categories of semantics are topological:

- **Kr**: $\mathbf{Kr}_X = \wp(X \times X)$; pullback along $f : X \to Y$,

$$(x, x') \in f^* R \iff (fx, fx') \in R, \text{ or } f^* R = f^\dagger \circ R \circ f.$$

- **Top**: $\mathbf{Top}_X$ = topologies with *reverse* inclusion; pullback,

$$U \in f^* \tau \iff \exists V \in \tau.\, U = f^{-1}(V), \text{ or } f^* \tau = f^{-1}[\tau].$$

- **Nb**: $\mathbf{Nb}_X = \wp(X \times \wp(X))^{\mathrm{op}}$; pullback,

$$x f^* E U \iff \exists V.\, U = f^{-1}(V) \,\&\, fxEV, \text{ or } f^* E = f^{-1} \circ E \circ f.$$

- Similarly for various subcategories **Pre**, **Eqv**, **Mon**, etc.

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The Category of Algebraic Semantics

Recall the famous result by Tarski and Lindenbaum [9],

$$\mathbf{CABA}^{\mathrm{op}} \cong \mathbf{Set}.$$

- Every CABA is isomorphic to some $\wp(X)$;
- Every morphism between CABA is isomorphic to some f^{-1}.

We may then define the category $\mathbf{CABAO}$ (morally, $\mathbf{CABAO}^{\mathrm{op}}$):

- Objects are pairs (X, m) with $m : \wp(X) \to \wp(X)$;
- A morphism $f : (X, m) \to (Y, n)$ is a function $f : X \to Y$ that

$$f^{-1} \circ n \subseteq m \circ f^{-1}.$$

- If f strictly commutes with n and m, we say it is *algebraic*.

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Algebraic Semantics is also Topological

We may verify now that **CABAO** is also topological (over **Set**):

- Fibres over X are $\left(\wp(X)^{\wp(X)}\right)^{\mathrm{op}}$, with *reverse* point-wise order.
- The pushforward along $f\colon X \to Y$ is given by

$$f_! m = \forall_f \circ m \circ f^{-1},$$

  where we have

$$\exists_f \dashv f^{-1} \dashv \forall_f.$$

- Actually, we have **Nb** $\cong$ **CABAO**.

Digression: Quantifier as Adjoints

Due to Lawvere [8], all first-order logical operations (including *identity*) can be characterised as certain adjoints between fibres:

- For $R \subseteq X \times Y$, define $\exists_R, \forall_R : \wp(X) \to \wp(Y)$,

$$\exists_R(S) = \{ y \in Y \mid \exists x.\, xRy \,\&\, x \in S \},$$
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  They form an adjunction $\exists_R \dashv \forall_{R^\dagger}$.

- f is functional iff $\exists_{f^\dagger} = \forall_{f^\dagger}$, and in that case they are f^{-1},

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- For first-order logic we have

$$[\![\forall x \varphi(x,y)]\!] = \forall_{\pi_Y} [\![\varphi(x,y)]\!], \quad [\![\exists x \varphi(x,y)]\!] = \exists_{\pi_Y} [\![\varphi(x,y)]\!].$$

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Valuation is also Topological

For any fixed P set of propositional variables, we have the category **Evl** of valuations:

- Objects are (X, V) with $V : \mathrm{P} \to \wp(X)$ a valuation of P on X;
- $f : (X, V) \to (Y, W)$ is an **Evl**-morphism iff $V \subseteq f^{-1} \circ W$.

Furthermore, we have

- Fibres $\mathbf{Evl}_X$ are $\wp(X)^{\mathrm{P}}$.
- Pullbacks are given by post-composition with f^{-1}.

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Products of Topological Categories

Definition

For any family of topological categories $\{\mathcal{A}_i\}_{i\in I}$, their *product*, denoted as $\prod_{i\in I}\mathcal{A}_i$, is given by

$$\left(\prod_{i\in I}\mathcal{A}_i\right)_X := \prod_{i\in I}(\mathcal{A}_i)_X.$$

This is indeed their categorical product (in some meta-category).

To this end, the complete description of semantics of modal logic is given by the *product* $\mathcal{A}\times\mathbf{EvI}$ of a modal category with $\mathbf{EvI}$.

The product construction also formalises multi-agency $\mathcal{A}^{\Sigma}$ (objects are $(X, (A_a)_{a\in\Sigma})$), or combine different types of modalities.

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Definition

A *modal category* is a topological category $(\mathcal{A}, |-|)$ equipped with a concrete functor $(-)^{+}$ to **CABAO**.

Any modal category provides interpretation for the modality:

$$[\![\Box\varphi]\!]_A := A^{+}[\![\varphi]\!]_A.$$

Familiar examples *embeds* into **CABAO**:

- **Kr**: $R \mapsto \Box_R$, $\Box_R = \forall_{R^\dagger}$.
- **Top**: $\tau \mapsto \Box_\tau$, $\Box_\tau$ is the interior operator.
- **Nb**: $E \mapsto \Box_E$, $\Box_E(S) = \{\, x \mid xES \,\}$.

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A Larger Picture

There is a well-behaved 2-category $\mathfrak{Topc}$ of topological categories,

$$
\begin{tikzcd}
 & \mathbf{Eqv} \arrow[d, hook] & \\
 & \mathbf{Pre} \arrow[u, bend left] \arrow[u, bend right] \arrow[dl, hook, bend left] \arrow[dr, hook, bend right] & \\
\mathbf{Kr} \arrow[ur, bend left] \arrow[dr, hook, bend right] \arrow[rr, phantom, "\dashv"] & & \mathbf{Top} \arrow[ul, bend right] \arrow[dl, hook, bend left] \\
 & \mathbf{LMon} \arrow[ul, bend right] \arrow[ur, bend left] \arrow[d, hook, bend right] & \\
 & \mathbf{Mon} \cong \mathbf{Evi} \arrow[u, bend right] \arrow[d, hook, bend right] \arrow[u, phantom, "\dashv"] & \\
 & \mathbf{Nb} \cong \mathbf{CABAO} \arrow[u, bend right] \arrow[u, phantom, "\dashv"] &
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Dependence and Group Structure

A General Word on Reasoning Patterns as Universal Structures

Different modal reasoning patterns now corresponds to universal structures within topological categories:

Modal Dependence	$\leftrightarrow$	Partial Orders within Fibres
Group Structure	$\leftrightarrow$	Lattice Operations within fibres
Logical Dynamics	$\leftrightarrow$	Connections between fibres

Theorem

Model transformation F preserves the interpretation of reasoning pattern A $\iff$[2] *F commutes with the semantic structure A corresponds to.*

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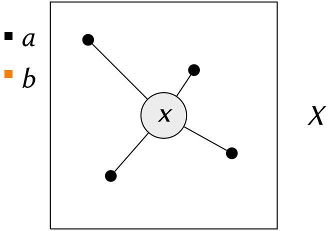
Modal Dependence

The order in fibres signifies the *epistemic strength* of each agent:



This signifies *dependence* in two agents' knowledge,

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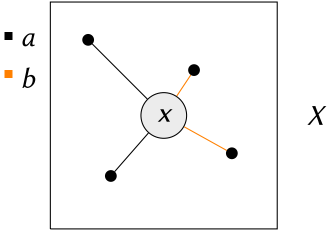

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A Local Order on CABAO

Order in $\mathcal{A}_X$ is *global*, while we need a *local* version of dependence:

Definition

For any two operators m, m' on $\wp(X)$ and for any $U \subseteq X$, we define a local relation $m \subseteq_U m'$ as for any $S \subseteq X$ and any $x \in U$,

$$x \in m(S) \Rightarrow x \in m'(S).$$

In other words, $m \subseteq_U m'$ iff $U \cap m(-) \subseteq U \cap m'(-)$ globally.

Fact

Let $\{U_i\}_{i \in I}$ be a family of subsets of X, then

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Interpretation of Dependence Atoms

Augment the language with *dependence atoms* $K_a b$ with $a, b \in \Sigma$.

For any modal category $(\mathcal{A}, (-)^+)$ and any $A \in \mathcal{A}$, we define

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[\![K_a b]\!]_{(A_a)_{a \in \Sigma}} &= \text{ the maximal subset } U \text{ of } X \text{ that } A_b^+ \subseteq_U A_a^+ \text{ holds;} \\
&= \{\, x \mid A_b^+ \subseteq_x A_a^+ \,\}.
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Or locally, for any $x \in X$ we have

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Previous general definition in different contexts:

- In **Kr**, $x \models K_a b$ iff $R_a[x] \subseteq R_b[x]$.
- In **Top**, $x \models K_a b$ iff $\mathrm{id} : (X, \tau_a) \to (X, \tau_b)$ is continuous at x.
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Group Structure

Group structure is about ways of combining agents' information.

Mathmatically, this amounts to finding functors of type

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And we have two canonical ones:

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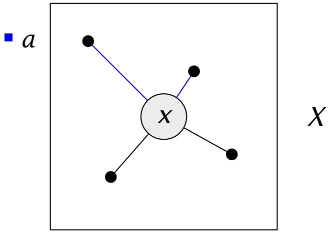
Canonical Group Structures



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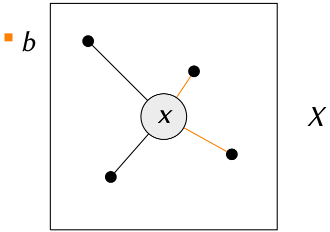

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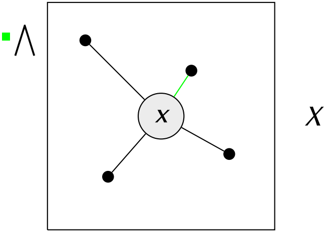

$\bigwedge_{a \in G} A_a$ is the universal agent that knows each individual knows.

It collects agents' *possitive* information: distributive knowledge.

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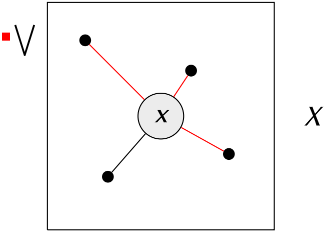

$\bigvee_{a \in G} A_a$ is the universal agent that it knows then each agent knows.

It collects agents' *negative* information: common knolwedge.

Some Examples

Group agents in different contexts:

- In **Kr**, we only have $\bigcap$ and $\bigcup$.
- In **Pre**, $\bigwedge$ is $\bigcap$, but $\bigvee$ is $\bigcup^*$.
- In **Top**, $\bigwedge$ is the topology generated by the subbase of the union of all topologies; $\bigvee$ is the intersection of topologies.
- In **Nb**, $\bigwedge$ is $\bigcup$ and $\bigvee$ is $\bigcap$.
- ...

Some Remarks

- Group structure depends on the ambient category.
- Groups as universal agents easily generalises.
- As long as $(\mathcal{A}, (-)^+)$ induces monotone and idempotent modalities, "common-knowledge-like" behaviors would appear:

$$\left(\bigvee_{a \in G} A_a\right)^+ \subseteq A_{a_1}^+ \circ \cdots \circ A_{a_n}^+, \ \forall a_1, \cdots, a_n \in G.$$

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- If we allow both dependence and forming groups, the following axioms would be valid in any modal category for $\bigwedge$-groups:
  - *Inclusion*: $K_G H$, provided $H \subseteq G$;
  - *Additivity*: $K_G H \wedge K_G P \to K_G(H \cup P)$;
  - *Transitivity*: $K_G H \wedge K_H P \to K_G P$;
  - *Transfer*: $K_G H \wedge \Box_H \varphi \to \Box_G \varphi$.

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- And similarly for $\bigvee$-groups:
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Logical Dynamics

Logical dynamics in general:

- Dynamical operator specifies *model change*.
- *Lift* geometric data to new model and interpret formulas there.
- Transfer the interpretation back to the *original* model.

In each clause, it is important that the original and updated models are related by some types of functions.

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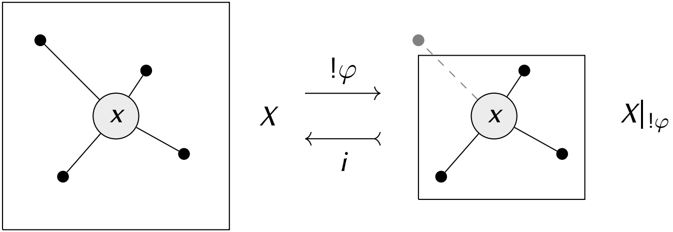
PAL Dynamics — Inclusion Model Change

- Announcing φ restricts the domain:



This amounts to an inclusion map $i : X|_{!\varphi} \hookrightarrow X$.

- New models are defined by pullback along i,

$$(A, V) \mapsto (i^* A, i^* V).$$

- Full interpretation uses $\exists_i$, $\forall_i$ to go back:

$$[\![[!\phi]\psi]\!]_A^V := \forall_i [\![\psi]\!]_{i^*A}^{i^*V}, \quad [\![\langle !\phi \rangle \psi]\!]_A^V := \exists_i [\![\psi]\!]_{i^*A}^{i^*V}.$$

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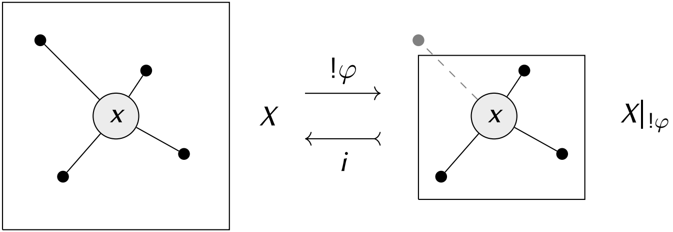

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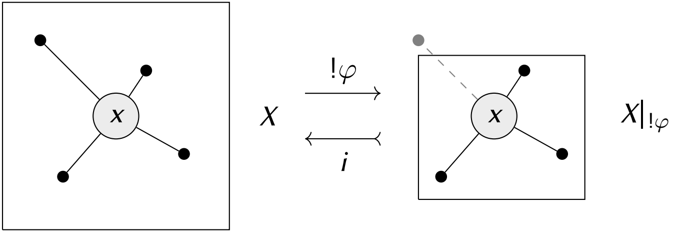

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Recursion axioms now becomes formal calculation of adjunctions:[3]

- For $p \in \mathrm{P}$,

$$[\![[!\phi]p]\!]_A^V = \forall_i [\![p]\!]_{i^*A}^{i^*V} = \forall_i \circ i^{-1} [\![p]\!]_A^V = [\![\phi \to p]\!]_A^V.$$

- For conjunction,

$$\begin{aligned} [\![[!\phi](\psi \wedge \chi)]\!]_A^V &= \forall_i ([\![\psi]\!]_{i^*A}^{i^*V} \cap [\![\chi]\!]_{i^*A}^{i^*V}) \\ &= \forall_i [\![\psi]\!]_{i^*A}^{i^*V} \cap \forall_i [\![\chi]\!]_{i^*A}^{i^*V} \\ &= [\![[!\phi]\psi \wedge \langle !\phi \rangle \chi]\!]_A^V. \end{aligned}$$

Recursion Axioms ii

- For negation, $i^{-1} \circ \forall_i = \mathrm{id} \Rightarrow \neg = \neg \circ i^{-1} \circ \forall_i = i^{-1} \circ \neg \circ \forall_i$,

$$\begin{aligned} [\![[!\phi]\neg\psi]\!]^V_A &= \forall_i \circ \neg [\![\psi]\!]^{i^* V}_{i^* A} \\ &= \forall_i \circ i^{-1} \circ \neg \circ \forall_i [\![\psi]\!]^{i^* V}_{i^* A} \\ &= [\![\phi \to \neg [!\phi]\psi]\!]^V_A. \end{aligned}$$

- For modality, in **Kr** and **Top** we know $\Box_{i^* A} = i^{-1} \circ \Box_A \circ \forall_A$,

$$\begin{aligned} [\![[!\phi] \Box \psi]\!]^V_A &= \forall_i \circ \Box_{i^* A} [\![\psi]\!]^{i^* V}_{i^* A} \\ &= \forall_i \circ i^{-1} \circ \Box_A \circ \forall_A [\![\psi]\!]^{i^* V}_{i^* A} \\ &= [\![\phi \to \Box [!\phi]\psi]\!]^V_A \end{aligned}$$

[3] This is essentially due to [6], which greatly inspires this project.

Some Remarks

- DEL can also be generalised to any modal category.
- What about other types of functional relations between the updated and the original models (new dynamics)?
- Recursion axioms only rely on formal properties of adjunctions.

Outlooks

Take a Step Back

In this work we made an essential use of **Set** and **CABAO**, which reflects our general choice of technical generality:

- Only consider basic modal logic.
- Interpret formulas as *subsets* of a model.

But our general methods and observations are by no means confined by such technical choices.

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The followings are kind of clear:

- We can consider general modal similarity types T.
- We can similarly define the category $\mathbf{CABAO}_T$.
- $\mathbf{CABAO}_T$ would still be topological over **Set**.

All of our constructions stated here can be generalised to this case.

Stone Space as Base Category

The choice of **Set** is based on the following assumption:

- Formulas are represented by *subsets* of models, and the Boolean operations are interpreted in power set algebras.

However, due to the finitary syntax of modal logic, we might want to consider interpreting formulas within general Boolean algebras.

Recall that $\mathbf{Set}^{\mathrm{op}} \cong \mathbf{CABA}$, hence the natural base category in this case is simply **Stone**, since again we have $\mathbf{Stone}^{\mathrm{op}} \cong \mathbf{BA}$.

The overall geometric picture would be similar, but we might need more general categorical structures than complete lattice fibres.

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Comparison with Coalgebras

The biggest different between our approach with the coalgebraic approach of modal logic is the choice of morphisms:

- The coalgebraic morphisms only correspond to *p-morphisms*, while we use the more general continuous maps, which are arguably more natural in exemplar semantics **Kr**, **Top**, **Nb**, etc.
- Our previous treatment on dependence, group agency and dynamics showed continuous maps are indispensable.
- We can actually *recover* p-morphisms by saying the semantic functor $(-)^+$ maps them to *algebraic* maps.

Comparison with Coalgebras — Continued

But there are also connections:

- If the set functor $\Omega \subseteq \wp \circ F$ is a subfunctor of $\wp \circ F$ for some F, we could define continuous morphisms between Ω-coalgebras as functions that satisfy certain lax conditions,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "\beta"] \\ \Omega X \arrow[r, "\Omega f"'] & \Omega Y \end{tikzcd}$$

- In good cases, Ω-coalgebras with continuous morphisms will be topological over **Set** (like **Kr** and **Nb**).
- Change our base to **Stone** would also subsume many categories of Stone coalgebras (cf. [7]).

Summary

What we have done:

- Unified semantics of modal logic using topological categories.
- Identified reasoning patterns $\leftrightarrow$ universal structures.
- Obtained syntactic-semantic correspondence for modal logic.
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