Categorical Structure in Theory of Arithmetic

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Introduction

Let $\mathbb{T}$ be some sufficiently strong theory of arithmetic. A formula is Σ_1 if it is provably equivalent to a coherent formula ($\top, \wedge, \bot, \vee, \exists$).

Proposition

A subset of $\mathbb{N}$ is r.e. iff it is definable by a Σ_1-formula.

A function $f : \mathbb{N}^k \to \mathbb{N}$ is *provably total (recursive)* in $\mathbb{T}$ if:

- There is a Σ_1-formula $\varphi_f(\bar{x}, y)$ defining the graph of f;
- $\mathbb{T} \vdash \forall \bar{x} \exists_! y \varphi_f(\bar{x}, y)$.

This class will be denoted as $\mathfrak{R}(\mathbb{T})$. It measures the strength of $\mathbb{T}$.

Complexity and Arithmetic

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Complexity and Arithmetic — Cont.

Logicians have considered a wide variety of arithmetic theories,

- PA, $I\Sigma_n$, EA, PA^-, Q, S_n^k, ...

When $\mathbb{T}$ is $I\Sigma_1$ (PA but with induction restricted to Σ_1-formulas):

Theorem (⋆)

Provably total functions in $I\Sigma_1$ are exactly p.r. functions.

+ Another equivalent way of characterising p.r. functions.
+ $\mathfrak{R}(\mathbb{T})$ is intimately related to the proof-theoretic ordinal of $\mathbb{T}$.
- Most/All proofs are like "programs on machine code".

We intend to provide a *structural* understanding of (⋆).

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Coherent logic is the fragment of first-order logic with:

- Formulas built up from $\top, \wedge, \bot, \vee, \exists$;
- Proofs formulated in sequent style $\varphi \vdash_{\overline{x}} \psi$;

A coherent theory $\mathbb{T}$ is *classical* if every formula $\varphi(\overline{x})$ has *complement* $\widetilde{\varphi}(\overline{x})$, such that $\mathbb{T}$ proves

$$\top \vdash_{\overline{x}} \varphi(\overline{x}) \vee \widetilde{\varphi}(\overline{x}), \quad \varphi(\overline{x}) \wedge \widetilde{\varphi}(\overline{x}) \vdash_{\overline{x}} \bot.$$

In particular if $\mathbb{T}$ contains negation with excluded middle.

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Categorical Logic — Cont.

Any $\mathbb{T}$ has a *syntactic category* $\mathcal{C}[\mathbb{T}]$ encapsulating itself:

- Objects are formulas (with contexts) in $\mathbb{T} / \sim_\alpha$;
- Morphisms $\theta : \varphi(\overline{x}) \to \psi(\overline{y})$ are $\mathbb{T}$-functional formulas / $\sim_\mathbb{T}$:

$$\theta(\overline{x}, \overline{y}) \vdash_{\overline{x},\overline{y}} \varphi(\overline{x}) \wedge \psi(\overline{y})$$

$$\varphi(\overline{x}) \vdash_{\overline{x}} \exists \overline{y} \theta(\overline{x}, \overline{y})$$

$$\theta(\overline{x}, \overline{y}) \wedge \theta(\overline{x}, \overline{z}) \vdash_{\overline{x},\overline{y},\overline{z}} \overline{y} = \overline{z}$$

Functorial Semantics

Sending a model M to a functor $\varphi \mapsto [\![\varphi]\!]_M$ gives an equivalence

$$\mathbf{Coh}(\mathcal{C}[\mathbb{T}], \mathbf{Set}) \simeq \mathrm{Mod}(\mathbb{T}).$$

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A Coherent Theory of Arithmetic

We want to find a suitable *coherent* theory of arithmetic $\mathbb{T}$ that faithfully represents the relevant fragment of $I\Sigma_1$:

- All formulas are Σ_1, with provable total functions correctly encoded as certain morphisms in $\mathcal{C}[\mathbb{T}]$.
- It contains PA^- (possibly more) plus a version of Σ_1-induction.

The construction of $\mathbb{T}$ should result in the following theorem:

Theorem (Correctness)

The interpretation of $\mathbb{T}$ into $I\Sigma_1$ induces an equivalence

$$\mathcal{C}[\mathbb{T}] \simeq \mathcal{C}[I\Sigma_1]_{\Sigma_1},$$

where $\mathcal{C}[I\Sigma_1]$ is the full subcategory of Σ_1-fragment of $I\Sigma_1$.

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A Coherent Theory of Arithmetic — Cont.

Given such $\mathbb{T}$, the *subject* of $(\star)$ can be easily recognised in $\mathcal{C}[\mathbb{T}]$:

- Let $[n]$ denote $\bigwedge_{1 \leq i \leq n} x_i = x_i$. We think of $[1]$ as the natural numbers in $\mathcal{C}[\mathbb{T}]$, with $[n] \cong [1]^n$ in $\mathcal{C}[\mathbb{T}]$.

Observation

$\mathcal{C}[\mathbb{T}]([n], [1])$ corresponds to provably total functions of $\mathbb{T}$ ($I\Sigma_1$).

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Strategy

According to categorical logic, the standard model $\mathbb{N}$ induces:

$$\mathcal{C}[\mathbb{T}] \xrightarrow{N} \mathbf{Set}$$

N maps every $\theta : [n] \to [1]$ to the function $\mathbb{N}^n \to \mathbb{N}$ it defines. The hard part of $(\star)$ is to show the images of these morphisms are p.r.

Strategy

$(\star)$ now is equivalent to the existence of a factorisation:

$$\begin{tikzcd} \mathcal{C}[\mathbb{T}] \arrow[rr, "N"] \arrow[dr, dashed] & & \mathbf{Set} \\ & \mathbf{PriM} \arrow[ur, hook] & \end{tikzcd}$$

where **PriM** morally is a category with

- Objects being r.e. subsets of $\mathbb{N}^n$;
- Morphisms being p.r. functions.

Strategy — Cont.

Such a situation begs for *initiality* result: $\mathcal{C}[\mathbb{T}]$ should be initial among certain class of categories containing **PriM** and **Set**.

Theorem (Initiality)

$\mathcal{C}[\mathbb{T}]$ *is initial w.r.t. coherent categories with a PNO.*

Examples of coherent categories with a PNO:

- **Set**, **PriM**, any topos with a NNO ...

Now $(\star)$ is implied by **Correctness** + **Initiality**.

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Coherent Theory of Arithmetic

The design of $\mathbb{T}$ should take into account the following points:

- **Validity**: What's present in $\mathbb{T}$ should be universally valid in all coherent categories with PNO, and preserved by such functors.
- **Strength**: $\mathbb{T}$ should be strong enough for $\mathcal{C}[\mathbb{T}]$ to have a PNO.

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Construction of Coherent Arithmetic

We construct $\mathbb{T}$ as follows:

- It has a constant 0.
- It has all primitive function names as function symbols, plus their corresponding defining axioms.
- Besides coherent logic, it has an induction rule:

$$\frac{\varphi(\overline{x}) \vdash_{\overline{x}} \psi(\overline{x}, 0) \qquad \varphi(\overline{x}) \wedge \psi(\overline{x}, y) \vdash_{\overline{x}, y} \psi(\overline{x}, sy)}{\varphi(\overline{x}) \vdash_{\overline{x}, y} \psi(\overline{x}, y)}$$

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Construction of Coherent Arithmetic — Cont.

The primitive function names *PR* and corresponding axioms:

- $\mathrm{z}, \mathrm{s}, \pi_n^k \in PR$, with corresponding axioms

$$\top \vdash_x \mathrm{z}x = 0, \quad \top \vdash_{x_1,\cdots,x_n} \pi_n^k(x_1, \cdots, x_n) = x_k.$$

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- $\mathrm{h}, \mathrm{g}_1, \cdots, \mathrm{g}_n \in PR$ with correct arity, $\mathrm{Cn}[\mathrm{h}, \mathrm{g}_1, \cdots, \mathrm{g}_n] \in PR$,

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Proof of Initiality

Parametrised Natural Number Object

In a Cartesian category $\mathcal{C}$, an object N is a PNO if we have

$$1 \xrightarrow{0} N \xleftarrow{s} N$$

such that for any $g : A \to B$ and $h : A \times N \times B \to B$, there is a *unique* map $rec_{g,h} : A \times N \to B$ making the following commute,

$$\begin{tikzcd}
A \arrow[r, "\langle \mathrm{id}, 0 \rangle"] \arrow[dr, "g"'] & A \times N \arrow[d, "rec_{g,h}"] & A \times N \arrow[l, "\mathrm{id} \times s"'] \arrow[d, "\langle \mathrm{id}, rec_{g,h} \rangle"] \\
& B & A \times N \times B \arrow[l, "h"]
\end{tikzcd}$$

Theorem

For a PNO N in $\mathcal{C}$, there is a unique map $ev : PR \to Mor(\mathcal{C})$, which is preserved by Cartesian functors preserving the PNO.

Proof.

Consider the following diagramme:

$$\begin{tikzcd}
N^n \arrow[r, "\langle \mathrm{id}, 0 \rangle"] \arrow[dr, "g"'] & N^n \times N \arrow[d, "rec_{g,h}"] & N^n \times N \arrow[l, "\mathrm{id} \times s"'] \arrow[d, "\langle \mathrm{id}, rec_{g,h} \rangle"] \\
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$\square$

Induction Principle of PNO

Theorem

The induction rule is valid for a PNO: For any object X, if

$$X \vDash \varphi(x) \vdash \psi(x,0) \quad X \times N \vDash \varphi(x) \wedge \psi(x,n) \vdash \psi(x,sn),$$

then we also have

$$X \times N \vDash \varphi(x) \vdash \psi(x,n).$$

Proof.

Take the usual proof of induction of an NNO to the parametrised version. □

Together they have shown **Validity**.

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PNO in $\mathcal{C}[\mathbb{T}]$

The remaining work is to show $[1] =: N$ is a PNO in $\mathcal{C}[\mathbb{T}]$:

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\varphi \arrow[r, "\langle \mathrm{id}, 0 \rangle"] \arrow[dr, "\gamma"'] & \varphi \times N \arrow[d, dashed, "rec_{\gamma,\theta}"] & \varphi \times N \arrow[l, "\mathrm{id} \times s"'] \arrow[d, dashed, "rec_{\gamma,\theta}"] \\
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This requires us to show we can encode finite lists of numbers in $\mathbb{T}$:

$$rec_{\gamma,\theta}(x, n, y) := \exists l(|l| = sn \wedge \gamma(x, l_0) \wedge \forall u{<}n\, \theta(l_u, l_{su}) \wedge l_n = y).$$

This is standard in arithmetic.

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This is standard in meta-logic practice.

The Lie I've been Telling

Σ_1-formulas of $I\Sigma_1$ also allow *bounded* universal quantification:

- For the above construction to work, we also requires bounded universal quantifiers in $\mathbb{T}$, and the actual $\mathbb{T}$ has them.
- For our proof to work, we further need to show **Validity** for them. This can be done in a *cohernet* setting.
- Using this, we can show **Strength**, and conclude **Initiality**.

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- It is a classical result in topos theory that classical logic is *conservative* over the coherent fragment.
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Further Application of Initiality

Glueing Argument

Let $\Gamma : \mathcal{C}[\mathbb{T}] \to \mathbf{Set}$ be the global section, sending a formula to its set of provable points. On $\mathsf{Sub}(1)$, it acts like *provability predicate.*

We can then glue $\mathcal{C}[\mathbb{T}]$ along Γ,

$$\begin{tikzcd} \widehat{\mathcal{C}[\mathbb{T}]} \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\downarrow"] & \mathbf{Set} \arrow[d, equal] \\ \mathcal{C}[\mathbb{T}] \arrow[r, "\Gamma"'] & \mathbf{Set} \end{tikzcd}$$

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Initiality of $\mathcal{C}[\mathbb{T}]$ implies there is a functor R satisfying

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In particular, R is coherent and preserves the PNO:

$$R\varphi = (\varphi, \alpha_\varphi : N\varphi \to \Gamma\varphi).$$

- $N = qR : \mathcal{C}[\mathbb{T}] \to \mathbf{Set}$ is also coherent and preserving PNO, hence it is the semantic functor N we constructed previously.
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Constructive Features of $\mathbb{T}$

Theorem

A Σ_1-sentence is provable in $I\Sigma_1$ iff it is true.

Proof.

The algorithm is provided by α. $\square$

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What we have done:

- A *structural understanding* of $(\star)$ by showing the Σ_1-fragment of $I\Sigma_1$ presents the initial coherent category with a PNO.
- Initiality proof of various constructive features of $\mathbb{T}$.

In the future:

- A pure categorical analysis of incompleteness.
- Extend to other theories, which requires a categorical treatment of various small ordinals.

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Thanks for Listening!

Appendix: Construction of PriM

It is not entirely clear what is a category of r.e. sets with p.r. functions. The key fact is the following proposition:

Theorem

Every r.e. set has a p.r. enumeration.

We construct the category **PriM** as follows:

- Objects are either $\varnothing$, or (S, s) where S is r.e. and $s : \mathbb{N} \to S$ is a p.r. enumeration of S.
- Maps are pairs $(f, \tilde{f})$, where $f : S \to T$ is a function tracked by a p.r. $\tilde{f}$ acting on codes,

$$\begin{tikzcd} \mathbb{N} \arrow[r, two heads, "s"] \arrow[d, "\tilde{f}"'] & S \arrow[d, "f"] \\ \mathbb{N} \arrow[r, two heads, "t"'] & T \end{tikzcd}$$

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